A Tour of Reflections Groups, Upside Down

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1 Upside Down?

Open virtually any standard textbook on Lie algebras or representation theory (e.g. [Hum90], [Hum72]), and you will encounter an exposition that looks something like this:

- Fix a Euclidean space $E = \mathbb{R}^n$ equipped with the standard positive-definite inner product $(-,-)$.
- Define a root system $\Phi \subseteq E$ as a finite set of vectors satisfying geometric reflection symmetries.
- Define the **coroot** $\alpha^\vee$ by pulling a formula out of a hat:

$$\alpha^\vee := \frac{2\alpha}{\|\alpha\|^2}$$

- Demand the "integrality condition": $\langle \beta, \alpha^\vee \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.
- Construct the Weyl group W, classify root systems, observe that some roots are longer than others, and finally—often as an afterthought—abstract away the ambient space to introduce Coxeter groups and Tits' geometric representation.

If the book is about the geometric representation of Coxeter groups (e.g. [AF05]), the approach might looks like:

- Start with a given Coxeter matrix (m_{ij}).
- Make a vector space V equipped with a symmetric bilinear form B defined on the standard basis $(e_i)_{1\le i\le n}$ via $B(e_i, e_j) = -\cos(\pi/m_{ij})$.
- Claim that the reflections $s_i(v) = v - 2B(v, e_i)e_i$ gives a representation of the Coxeter group.

This article concerns the following problems arising from the above expositions:

- For the first exposition, one starts with a fully formed Euclidean space, with notions of metrics, lengths, and angles. However, it turns out that in the end we dislike the ambient metric and want to abstract it away: all computational tricks suggest that instead of a pure inner product (v, α), we should prefer vectors paired with those $\vee$-ed vectors like $\langle v, \alpha^\vee \rangle$, where a mysterious normalization $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ is introduced to make computation smooth. **No underlying reason for this computational trick is ever explained.**

- The above two expositions are both **problem-oriented**: Root systems arise in the context of Lie algebras and is of its independent intrest; expressing the Coxeter group as a reflection group does not require prior knowledge of root systems and is often presented independently. Despite their similarities being the elephant in the room, **little effort is made to unify them in the literature.**

- Deeper study of algebraic groups requires a more general notion of root systems, i.e. the so-called *root-datum*. Few expositions explain how to connect the dots between root systems and root data, and how to generalize the former to the latter. (Unfortunately there are historical reasons: the axiomization of root systems was done almost the same time as the development of algebraic groups.)

To address these problems, we must turn the exposition upside down. We instead adopt a **structure-oriented approach: Never introduce any additional structure until you are backed into a corner and forced to do so.** No ambient metric. No lengths. No angles. We start with nothing but a blank vector space and linear duality, and we will watch both Coxeter groups and root systems construct themselves canonically.

2 Reflections

Let V be a finite-dimensional real vector space. Let $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ be its dual space. The only canonical operation we possess is the natural pairing between a vector and a functional:

$$\langle v, f \rangle := f(v) \quad (v \in V,\, f \in V^*)$$

We want to define a linear reflection $s : V \to V$ without reference to "perpendicularity" or "distance". The following observation captures the required essence: a reflection fix a hyperplane (a mirror) and invert a direction transverse to it.

- **The Mirror:** The zero-set of a linear functional $\alpha^\vee \in V^*$:
$$H = \ker(\alpha^\vee) = \{v \in V \mid \langle v, \alpha^\vee \rangle = 0\}$$
- **The Reflection Line:** Spanned by a non-zero vector $\alpha \in V$.

Note that α and $\alpha^\vee$ are currently completely unrelated. A reflection given by the pair $(\alpha, \alpha^\vee)$ is a linear map $s : V \to V$ that

- fixes H pointwise: $s(v) = v$ for all $v \in H$,
- inverts α: $s(\alpha) = -\alpha$.

If $\alpha \in H$, or equivalently $\langle \alpha, \alpha^\vee \rangle = 0$, then α is forced to be zero, so let's rule that out. As long as $\alpha \notin H$, above conditions uniquely determine the reflection s by the pair $(\alpha, \alpha^\vee)$.

You must want a explicit formula for s. So let's say $v \in V$ is arbitrary with decomposition $v = h + c\alpha$ for some $h \in H$ and $c \in \mathbb{R}$. Then:

$$s(v) = s(h + c\alpha) = h - c\alpha = v - 2c\alpha$$

This c can be extracted from the pairing with $\alpha^\vee$:

$$\langle v, \alpha^\vee \rangle = \langle h + c\alpha, \alpha^\vee \rangle = c\langle \alpha, \alpha^\vee \rangle$$

Hence the explict reflection formula:

$$s(v) = v - 2\frac{\langle v, \alpha^\vee \rangle}{\langle \alpha, \alpha^\vee \rangle}\alpha$$

It's annoying to carry around the denominator $\langle \alpha, \alpha^\vee \rangle$. As reflections doesn't really care about the scale of α or $\alpha^\vee$, let's just require that $\langle \alpha, \alpha^\vee \rangle = 2$ to make the life easier. Then the reflection formula simplifies to:

$$s(v) = v - \langle v, \alpha^\vee \rangle\alpha$$

We call α a *root* and $\alpha^\vee$ a *coroot*.

2.1 Inner Product

To link back to the traditional definition of a root system, we need to introduce an inner product, though it is completely unnecessary for the most part of the theory. How to characterize a "well-behaved" inner product? The following gives a good answer:

Proposition 1 (well-behaveness of an inner product)**.** Let V be a vector space with an inner product $(\cdot,\cdot)$ (in fact, a symmetric bilinear form with $(\alpha,\alpha)\neq 0$ suffices). For a given reflection s defined by $(\alpha,\alpha^\vee)$ with $\langle\alpha,\alpha^\vee\rangle = 2$, TFAE:

1. the hyperplane $H = \ker(\alpha^\vee)$ is indeed perpendicular to α with respect to $(\cdot,\cdot)$, i.e. for all $v \in V$:
$$\langle v,\alpha^\vee\rangle = 0 \implies (v,\alpha) = 0$$
2. the reflection s given by $(\alpha,\alpha^\vee)$ is an isometry with respect to $(\cdot,\cdot)$, i.e. for all $v, w \in V$:
$$(s(v), s(w)) = (v, w)$$
3. The abstact pairing $\langle -,-\rangle$ and the concrete inner product $(\cdot,\cdot)$ admit the following compatibility:
$$\langle v,\alpha^\vee\rangle = \frac{2(v,\alpha)}{(\alpha,\alpha)}$$

Proof. First let's note a direct computation: for any $v, w \in V$,

$$\begin{aligned}(s(v), s(w)) &= (v - \langle v,\alpha^\vee\rangle\alpha, w - \langle w,\alpha^\vee\rangle\alpha) \\ &= (v,w) - \langle w,\alpha^\vee\rangle(v,\alpha) - \langle v,\alpha^\vee\rangle(\alpha,w) + \langle v,\alpha^\vee\rangle\langle w,\alpha^\vee\rangle(\alpha,\alpha)\end{aligned} \tag{1}$$

(1) $\implies$ (2): If H is perpendicular to α, then for any $v \in V$ with $\langle v,\alpha^\vee\rangle = 0$, we have $(v,\alpha) = 0$. Thus by Equation 1, $(s(v), s(w)) = (v, w)$ for any $w \in V$. As $\alpha \notin H = \ker(\alpha^\vee)$, by symmetry of $(\cdot,\cdot)$, it remains to check $(s(\alpha), s(\alpha)) = (\alpha,\alpha)$, which is trivial.

(2) $\implies$ (3): If s is an isometry, then by Equation 1, we have

$$\langle v,\alpha^\vee\rangle\langle w,\alpha^\vee\rangle(\alpha,\alpha) = \langle w,\alpha^\vee\rangle(v,\alpha) + \langle v,\alpha^\vee\rangle(\alpha,w)$$

take $w = \alpha$ and recall that $\langle\alpha,\alpha^\vee\rangle = 2$, we have

$$2\langle v,\alpha^\vee\rangle(\alpha,\alpha) = 2(v,\alpha) + \langle v,\alpha^\vee\rangle(\alpha,\alpha)$$

or equivalently

$$\langle v,\alpha^\vee\rangle = \frac{2(v,\alpha)}{(\alpha,\alpha)}$$

(3) $\implies$ (1): Clearly with the above equation, $(v,\alpha) = 0$ iff $\langle v,\alpha^\vee\rangle = 0$, which is exactly the perpendicularity condition. □

So any inner product $(\cdot,\cdot)$ satisfying the above equivalent conditions is considered "well-behaved". We pose extra emphasis on the last identity

$$\langle v,\alpha^\vee\rangle = \frac{2(v,\alpha)}{(\alpha,\alpha)} = \left(v, \frac{2}{(\alpha,\alpha)}\cdot\alpha\right)$$

Traditionally, the notation $\alpha \mapsto \alpha^\vee$ is viewed as a "normalization" with respect to the inner product $(\cdot,\cdot)$ to make the abstract pairing $\langle v,\alpha^\vee\rangle$ may be interpreted directly as the inner product $(v,\alpha^\vee)$ in V. The above identity validates this interpretation: define a "virtual" $\alpha^\vee \in V$ by

$$\alpha^\vee := \frac{2}{(\alpha,\alpha)}\cdot\alpha$$

does the job.

2.2 Root Length Ratio

With an inner product $(\cdot,\cdot)$, we may define the length of a root α. As it relies on a choosen inner product, it should not be considered a canonical invariant of a root system. However, the length ratios between the roots is an intrinsic property. To see this, recall the well-behaveness condition of the inner product $(\cdot,\cdot)$ from Proposition 1. Applying it to two different roots α_i and α_j, we have

$$a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} \qquad a_{ji} = \langle \alpha_i, \alpha_j^\vee \rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$$

hence

$$a_{ji} : a_{ij} = (\alpha_i,\alpha_i) : (\alpha_j,\alpha_j)$$

So if we exclude the orthogonal $\theta = \pi/2$ case, the ratio of the lengths of two roots is completely determined by the Cartan matrix, without ever introducing a metric.

3 More Reflections

Now let us consider a set of n mirrors, indexed by $I = \{1,\ldots,n\}$:

- *Simple roots*: $\Delta := (\alpha_i)_{i\in I} \subseteq V$
- *Simple coroots*: $\Delta^\vee := (\alpha_i^\vee)_{i\in I} \subseteq V^*$
- the normalization condition: $\langle \alpha_i, \alpha_i^\vee \rangle = 2$ for all i.

Each pair defines a *simple reflection*:

$$s_i(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i$$

How do two different reflections s_i and s_j interact? We define the **Cartan matrix** $A = (a_{ij})_{i,j\in I}$ by the mutual evaluations $a_{ij} := \langle \alpha_j, \alpha_i^\vee \rangle$.

- Its diagonal entries are already fixed: $a_{ii} = 2$.
- The off-diagonal entries dictate the action of the mirrors on each other's direction vectors: $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.

Remark. Recall that the natural pairing $\langle -,- \rangle$ is an application of a functional to a vector. So if you are suffering from the seemingly "reversed" indices in $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$, just write $a_{ij} = \alpha_i^\vee(\alpha_j)$: this is the right direction.

3.1 Cartan Matrix as a Bilinear Form

Note that **if Δ is a basis of V**, then $\Delta^\vee$ is determined by Δ and the Cartan matrix A by $\alpha_i^\vee(v) = \sum_{j\in I} a_{ij}\alpha_j^*(v)$, where $(\alpha_i^*)_{i\in I}$ is the dual basis of Δ. Thus usually we only need to specify a linearly independent collection Δ and the Cartan matrix A. To summarize, the Cartan matrix A should be viewed as a bilinear form $A : V \times V \to \mathbb{R}$ by

$$\begin{aligned} A(\alpha_i,\alpha_j) &:= a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle \\ A(\alpha_i,\cdot) &:= \alpha_i^\vee \in V^* \\ A(v,w) &:= \sum_{i,j\in I} a_{ij}\alpha_i^*(v)\alpha_j^*(w) \end{aligned}$$

3.2 Root Systems and Reflection Groups

The *reflection group* W generated by Δ is defined as the subgroup of $\mathrm{GL}(V)$ generated by the simple reflections $(s_i)_{i\in I}$. The *root system* Φ generated by Δ is defined as the orbit of Δ under the action of the reflection group W. Elements of Φ are called *roots*.

Completely analogously, we may define the *coroot system* $\Phi^\vee$ generated by $\Delta^\vee$ as the orbit of $\Delta^\vee$ under the action of W, where W acts on V^* via the natural contragradient representation $w \cdot f := f \circ w^{-1}$ for $w \in W$ and $f \in V^*$. Elements of $\Phi^\vee$ are called *coroots*.

When Δ is a basis of V, as the simple coroots are completely determined by the simple roots, we may just focus on the root system Φ.

Remark. **Note that our definition of root systems and coroot systems is very general. Indeed, contains just merely enough information to recover a reflection group with fixed simple reflections.** There are at least four layers of generality:

- Δ is not required to be a basis of V.
- The reflection group may not be a Coxeter group.
- The root system Φ could well be infinite.
- No crystallographic restriction is imposed.

TODO: Of course we shall reconnect this to the traditional axiomatic definition of a root system, i.e. the Humphreys one. This will be done later.

- A reference on systematic axiomization of general root systems: [GHP18]

4 Interaction Between Reflections

We now start considering the group-theoretic structure generated by these simple reflections, i.e. the reflection group W. By definition of a reflection, $s_i^2 = \mathrm{id}$ for all i. The next question is: what is the product $s_i s_j$, and what is its order?

The intuition is that $s_i s_j$ is a rotation in the plane spanned by α_i and α_j. But how to interpret this intuition without a metric? The key is the eigenvalues: it's the only canonical invariant of a linear map that does not require any additional structure. Let's compute $s_i s_j$ acting on a random vector v first:

$$\begin{aligned} s_i s_j(v) &= s_i(v - \langle v, \alpha_j^\vee \rangle \alpha_j) \\ &= v - \langle v, \alpha_i^\vee \rangle \alpha_i - \langle v, \alpha_j^\vee \rangle s_i(\alpha_j) \\ &= v - \langle v, \alpha_i^\vee \rangle \alpha_i - \langle v, \alpha_j^\vee \rangle (\alpha_j - a_{ij}\alpha_i) \\ &= v + \left(a_{ij}\langle v, \alpha_j^\vee \rangle - \langle v, \alpha_i^\vee \rangle\right) \alpha_i - \langle v, \alpha_j^\vee \rangle \alpha_j \end{aligned} \tag{2}$$

In paticular, $s_i s_j$ is stable on the plane $U := \mathrm{span}(\alpha_i, \alpha_j)$, with matrix representation

$$\begin{aligned} (s_i s_j)(\alpha_i, \alpha_j) &= (\alpha_i, \alpha_j) \begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix} \\ &= (\alpha_i, \alpha_j) \begin{pmatrix} a_{ij}a_{ji} - 1 & a_{ij} \\ -a_{ji} & -1 \end{pmatrix} \end{aligned} \tag{3}$$

Following the intuition, composing two reflections should give a "rotation" in the plane U. We formalize this via eigenvalues. Note that by elementary block-triangular interpretation of invariant subspaces applied to characteristic polynomials, the eigenvalues of $s_i s_j$ are exactly the disjoint union of:

- eigenvalues of $s_i s_j$ on the quotient space V/U;

- eigenvalues of $s_i s_j$ restricted to U.

For the first part, by Equation 2, $s_i s_j$ acts as the identity on V/U, so all eigenvalues are 1. The second part given by Equation 3 is more interesting. Intuitively, if the angle between α_i and α_j is θ, then we expect a 2-dimensional rotation matrix of the form:

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

But as α_i and α_j are not "orthogonal" in any sense, we cannot directly read off the rotaion angle 2θ from the matrix entries. Even θ itself is not well-defined, as we have not yet introduced a metric whatsoever. So, what is θ?

It lies in the argument of the eigenvalues of $s_i s_j$ restricted to U. Let's compute them. Via Equation 3, **if α_i and α_j are linearly independent**, the characteristic polynomial of $s_i s_j$ restricted to U is given by

$$\lambda^2 - (a_{ij}a_{ji} - 2)\lambda + 1$$

so the eigenvalues, λ and $\overline{\lambda}$, are of norm 1 and of real part $\frac{1}{2}(a_{ij}a_{ji}-2)$. Let $\lambda := e^{i\cdot 2\theta}$, then we have $\cos 2\theta = \frac{1}{2}(a_{ij}a_{ji} - 2)$. Cleaning up the equation a little by the standard trigonometric identity $\cos 2\theta = 2\cos^2\theta - 1$ gives the well-known identity:

$$a_{ij}a_{ji} = 4\cos^2\theta \tag{4}$$

So the eigenvalues of $s_i s_j$ are exactly $e^{\pm i\cdot 2\theta}$ 【TODO: if $0 < a_{ij}a_{ji} < 4$. We should be more careful here.】 , and the order of $s_i s_j$ is finite if and only if $\frac{\theta}{\pi} \in \mathbb{Q}$. In that case, the order m_{ij} of $s_i s_j$ is the reduced denominator of $\frac{\theta}{\pi}$. If $\frac{\theta}{\pi} \notin \mathbb{Q}$, then $s_i s_j$ has infinite order, and we set $m_{ij} = 0$ (or ∞, depending on your taste).

Remark. Despite being informal under our treatment, the identity $a_{ij}a_{ji} = 4\cos^2\theta$ may be obtained by comparing the trace of the rotation matrix

$$\operatorname{tr}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} = 2\cos 2\theta$$

with the trace of the matrix in Equation 3.

Finally, we consider a corner case: $\theta = 0$, or equivalently $a_{ij}a_{ji} = 4$. It makes the two mirrors "parallel". 【TODO: This case is nontrivial. Seemingly infinite dihedral groups arise. Need more care. Reducted.】

To summarize, we have already shown that the reflection group W generated by s_i is a quotient of the abstract Coxeter group defined by the presentation:

$$\langle s_1, \dots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle$$

where the order m_{ij} of $s_i s_j$ is solely determined by the product $a_{ij}a_{ji}$. As we said before, our definition of a root system is very general, whose reflection group may not be a Coxeter group. Nevertheless, the above shows that it is at least its quotient. **TODO: Analysis on when this quotient gives an isomorphism requires more work.**

5 The Divergence: Symmetric vs. Crystallographic

As we see from Equation 4, the product $a_{ij}a_{ji}$ is the only quantity that matters in determining the angle between the two mirrors, and hence the order m_{ij} of $s_i s_j$. So fixing m_{ij}, we have the freedom to choose any pair (a_{ij}, a_{ji}) that satisfies $a_{ij}a_{ji} = 4\cos^2(\pi/m_{ij})$. This freedom now diverges into two different directions, depending on what context we are in.

5.1 Geometric Representation: The Symmetric Choice

Suppose our goal is purely group-theoretic. We play with the reflections s_i on V because we want to construct an abstract group representation of the Coxeter group defined by the Coxeter matrix (m_{ij}). The most canonical, unbiased choice is to do it symmetrically:

$$a_{ij} = a_{ji} := -2\cos\left(\frac{\pi}{m_{ij}}\right)$$

where the negative sign is chosen so that reflections act as opposing walls of a convex chamber, a purely conventional choice. This is the *Geometric Representation of a Coxeter Group.* Once Δ is chosen to be a basis of V, it makes the Cartan bilinear form A symmetric.

TODO: We have not yet shown that this construction indeed gives a Coxeter group. This requires more work.

Recall Proposition 1. **After symmetrization, the Cartan bilinear form A itself is a "well-behaved" symmetric bilinear form**: the last identity in Proposition 1 may be verified:

$$\langle v, \alpha_i^\vee \rangle = A(v, \alpha_i) = \frac{2A(v, \alpha_i)}{A(\alpha_i, \alpha_i)}$$

It is now natural to further question whether A is positive-definite, so that V obtains a bona fide Euclidean structure canonically from the Coxeter matrix (m_{ij}). But before that, we introduce a shift of notation.

Remark. Traditionally the common practice is to set α_i to be unit vectors, and define a bilinear form B by $B(\alpha_i, \alpha_j) := -\cos(\pi/m_{ij})$ and with the reflections

$$s_i(v) = v - 2B(v, \alpha_i)\alpha_i$$

This is equivalent to the previous treatment via a direct rescaling of $A = 2B$. The reason for this choice may be the following:

- $B(\alpha_i, \alpha_i) = 1$, making the roots $\Phi := W \cdot \Delta$ all of unit length. As $B(\alpha_i, \alpha_i) = 1$ and B is invariant under reflections, all roots are on the unit sphere.
- In the crystallographic case we shall study later, we would like to reserve A for the Cartan matrix, which is an integer matrix. However, results in this subsection are still useful as sometimes a handy canonical choice of inner product is still desired. B is then used to denote the symmetrized version of A, solely for the purpose of equipping V with a canonical Euclidean structure.

For these reasons, We shall switch to above convention afterwards.

Proposition 2. The symmetric bilinear form B is positive-definite if and only if the Coxeter group is finite.

TODO: The proof of this proposition is beautiful, but it requires more work on root systems.

5.2 Crystallographic Root System: The Integral Choice

Recall that we let $a_{ij} = a_{ji}$ in $a_{ij}a_{ji} = 4\cos^2(\pi/m_{ij})$ in the previous treatment that finally gave us a the geometric representation of a Coxeter group. However, this is not the only choice. Another idea is the so-called *crystallographic restriction*: to force a_{ij} and a_{ji} to be integers, i.e. A is an integer matrix. Such a requirement is intresting because of the following TFAE characterization:

Proposition 3. TFAE for a root system Φ generated by a basis Δ, where Δ is a basis of V:

- The Cartan matrix A consists of integers only.
- The lattice $\mathbb{Z}\Delta$ is stable under the action of the reflection group W.
- $\mathbb{Z}\Delta = \mathbb{Z}\Phi$.

Proof. Trivial. $\square$

There aren't many choices for a_{ij} and a_{ji} actually. If $\theta \neq \pi/2$, with the restriction $0 < a_{ij}a_{ji} = 4\cos^2\theta < 4$, the only possible configurations are, respectively,

- The product $a_{ij}a_{ji}$: $1, 2, 3$,

- $\{|a_{ij}|, |a_{ji}|\}$: $\{1, 1\}, \{1, 2\}, \{1, 3\}$
- The root length ratios: $1 : 1, 1 : \sqrt{2}, 1 : \sqrt{3}$,
- The angle θ: $2\pi/3, 3\pi/4, 5\pi/6$ (if adopting all off-diagonal a_{ij} to be negative).

So in the crystallographic case, above four quantities are all determined by each other. Do note that this does not applies to $\theta = \pi/2$, where any root length ratio is possible.

6 Root Systems, Revisited

7 TODO

Severe problem: In the reflection interaction section, it's possible that $0 \leq a_{ij}a_{ji} \leq 4$ does not hold. In which case the solutions of the characteristic polynomial are REAL, hence being not a rotation. Turns out that being able to admit a well-behaved inner product is equivalent to the condition $0 \leq a_{ij}a_{ji} \leq 4$.

Meanwhile off-diagonal a_{ij} should be required to be non-positive, which is a must if we want $\Phi^+ \sqcup \Phi^- = \Phi$ (sign coherence). It is also said that sign coherence gives a Coxeter group. So to show that the geometric representation construction indeed gives a Coxeter group, we need to show that sign coherence is satisfied.

We should really try to explain the way to naturally land on the generalized Cartan matrix. A generalized Cartan matrix is a matrix $A = (a_{ij})$ satisfying:

- $a_{ii} = 2$ for all i,
- $a_{ij} \leq 0$ for all $i \neq j$,
- $a_{ij} = 0$ iff $a_{ji} = 0$,
- symmetrizable via a diagonal matrix by DA

Note that the last condition is equivalent to the existence of a well-behaved symmetric bilinear form, in our sense.

Kac-Moody root system seems to be relevant.

It's interesting to note that symmetrizing a Cartan matrix (in the crystallographic case) by DA and by DAD^{-1} are both extremely useful. The former gives a metric, while the latter is the geometric mean, which does not change the eigenvalues.

As the way that Mathlib takes, we should first define a root system, then a base and finally a Cartan matrix. Cartan matrix is based on a base.

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