Selected solutions to Atiyah-Macdonald's exercises

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Abstract

A non-comprehensive collection of solutions to Atiyah-Macdonald's exercises, with some additional comments and discussions. There may be errors, for reference only.

1 Chapter 2

Exercise 1 ([AM69]-exr-2.1)**.** $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ when m, n are coprime.

Proof. By the Chinese Remainder Theorem, there exists integers a, b such that

$$am + bn = 1$$

Then for any $x \otimes y \in \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$,

$$x \otimes y = 1 \cdot (x \otimes y) = (am + bn)(x \otimes y) = am(x \otimes y) + bn(x \otimes y) = 0 + 0 = 0.$$

Thus $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$. □

Exercise 2 ([AM69]-exr-2.3)**.** For A a local ring, M, N finitely generated A-modules. If $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

Proof. Tensoring with the residue field $k := A/\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal of A), we have

$$(M \otimes_A k) \otimes_k (N \otimes_A k) = 0$$

Since its a tensor product of vector spaces over field k, we must have $M \otimes_A k = 0$ or $N \otimes_A k = 0$, by dimension counting. Without loss of generality, say $M \otimes_A k = 0$. By Nakayama's lemma, this implies $M = 0$. □

Exercise 3 ([AM69]-exr-2.11)**.** Let A be a nonzero ring. Show that $A^m \cong A^n$ iff $m = n$.
Additionally, show that

- if $\varphi : A^m \to A^n$ is surjective, then $m \geq n$.
- If φ is injective, is it true that $m \leq n$?

Proof. By taking tensor product with $k := A/\mathfrak{m}$ (where $\mathfrak{m}$ is any maximal ideal of A, existence by non-zeroness) and counting dimensions as vector spaces. The first additional question by the right-exactness of tensor product. The second question seems rather hard [TODO]. □

Exercise 4 ([AM69]-exr-2.14: Direct limits)**.** Definition of direct limit of a direct system of A-modules $(M_i, \mu_{ij})_{i,j \in I}$ indexed by a directed set I. $M := C/D$ where C is the direct sum and D is the relations.

Proof. There's nothing to prove. □

Exercise 5 ([AM69]-exr-2.15)**.** Every element of $M := \varinjlim_i M_i$ is of the form $\mu_i(x_i)$ for some $i \in I$ and $x_i \in M_i$.

If $\mu_i(x_i) = 0$ in M, then $\exists j \geq i$ s.t. $\mu_{ij}(x_i) = 0$ in M_j.

Proof. Say $M \ni x = \sum_i \mu_i(x_i)$, where i runs over a finite subset of I and $x_i \in M_i$. By directedness of I, $\exists k \in I$ s.t. $k \geq i$ for all i in the finite subset. Then $\mu_k \left(\sum_i \mu_{ik}(x_i)\right) = x$.

If $\mu_i(x_i) = 0$ in M, then $x_i \in M_i \cap D$, that is, a finite sum in C

$$x_i = \sum_{j<k} y_{jk} - \mu_{jk}(y_{jk}) = \sum_k \left(\sum_{k<l} y_{kl} - \sum_{j<k} \mu_{jk}(y_{jk}) \right)$$

Comparing the summation componentwise in C gives

$$\begin{aligned} x_i &= \sum_{i<l} y_{il} - \sum_{j<i} \mu_{ji}(y_{ji}) \\ 0 &= \sum_{k<l} y_{kl} - \sum_{j<k} \mu_{jk}(y_{jk}) \quad \text{for all } k \neq i \end{aligned}$$

Let p be an upper bound of all indices appearing in the above equations. Apply μ_{ip} to the first equation and μ_{kp} to the second equation for all $k \neq i$. We get

$$\begin{aligned} \mu_{ip}(x_i) &= \sum_{i<l} \mu_{ip}(y_{il}) - \sum_{j<i} \mu_{jp}(y_{ji}) \\ 0 &= \sum_{k<l} \mu_{kp}(y_{kl}) - \sum_{j<k} \mu_{jp}(y_{jk}) \quad \text{for all } k \neq i \end{aligned}$$

Summing up all these equations gives $\mu_{ip}(x_i) = 0$. □

Exercise 6 ([AM69]-exr-2.16)**.** The universal property of direct limits.

Proof. "Up to isomorphism" part by abstract nonsense of category theory.

Say we have $\alpha_i : M_i \to N$ s.t. $\alpha_j \circ \mu_{ij} = \alpha_i$ for all $i \leq j$. Define $\varphi : C \to N$ by $\varphi \circ \mu_i := \alpha_i$. By the universal property of direct sum, such φ is well-defined. Also note that this is the only possible definition of φ satisfying $\varphi \circ \mu_i = \alpha_i$. This uniqueness also passes down to the quotient $M = C/D$. Say the quotient map is $\bar{\varphi} : M = C/D \to N$, it remains to show that $\bar{\varphi}$ is well-defined. For any generator of D, say $x_j - \mu_{jk}(x_j)$ for some $j \leq k$ and $x_j \in M_j$, we have $\varphi(x_j - \mu_{jk}(x_j)) = \alpha_j(x_j) - \alpha_k(\mu_{jk}(x_j)) = 0$. □

Exercise 7 ([AM69]-exr-2.17)**.** Any A-module is a direct limit of its finitely generated A-submodules.

Proof. It's clear that $\sum_i M_i = \bigcup_i M_i$ when the direct system is directed by inclusion. It's also easy to verify that $\bigcup_i M_i$ satisfies the universal property of direct limits. □

Exercise 8 ([AM69]-exr-2.18)**.** $\mathbf{M} := (M_i, \mu_{ij})$, $\mathbf{N} := (N_i, \nu_{ij})$ direct systems of A-modules indexed by the same directed set I. Let M, N be their direct limits and μ_i, ν_i be the canonical maps, respectively.

A homomorphism $\Phi : \mathbf{M} \to \mathbf{N}$ is defined to be a family of homomorphisms $\varphi_i : M_i \to N_i$ satisfying $\nu_{ij} \circ \varphi_i = \varphi_j \circ \mu_{ij}$ for all $i \leq j$. Show that Φ defines a unique homomorphism $\varphi = \varinjlim \Phi : M \to N$ such that $\varphi \circ \mu_i = \nu_i \circ \varphi_i$ for all i.

That is, taking direct limits is a functor from the category of direct systems of A-modules to the category of A-modules.

Proof. Let $\bar{\varphi}_i$ be the composition $\nu_i \circ \varphi_i : M_i \to N_i \to N$. Define our map $\varphi : M \to N$ by these maps, the required compatibility condition $\varphi \circ \mu_i = \nu_i \circ \varphi_i = \bar{\varphi}_i$ is guaranteed by hypothesis. □

Exercise 9 ([AM69]-exr-2.19)**.** Say $\mathbf{M} := (M_i)_{i\in I}$, $\mathbf{N} := (N_i)_{i\in I}$, $\mathbf{P} := (P_i)_{i\in I}$ are direct systems of A-modules indexed by the same directed set I. If the sequence of direct systems

$$\mathbf{M} \longrightarrow \mathbf{N} \longrightarrow \mathbf{P}$$

is exact, then the sequence of direct limits

$$M \longrightarrow N \longrightarrow P$$

is also exact. That is, taking direct limits is an exact functor from the category of direct systems of A-modules to the category of A-modules.

Proof. Say the maps in the direct systems are $\varphi_i : M_i \to N_i$ and $\psi_i : N_i \to P_i$. The maps in the direct limits are $\varphi : M \to N$ and $\psi : N \to P$ induced by φ_i and ψ_i respectively. The canonical maps are $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ and $\pi_i : P_i \to P$. These maps make the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \xrightarrow{\varphi_i} & N_i & \xrightarrow{\psi_i} & P_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_i} & & \downarrow{\scriptstyle \nu_i} & & \downarrow{\scriptstyle \pi_i} & & \\ 0 & \longrightarrow & M & \xrightarrow{\varphi} & N & \xrightarrow{\psi} & P & \longrightarrow & 0 \end{array}$$

Figure 1

We do diagram chasing. First, We have $(\psi \circ \varphi) \circ \mu_i = \psi \circ \nu_i \circ \varphi_i = (\pi_i \circ \psi_i) \circ \varphi_i = 0$. By the universal property of direct limits, $\psi \circ \varphi = 0$. Second, for any $y \in N$ with $\psi(y) = 0$, say $y = \nu_i(y_i)$ for some $i \in I$ and $y_i \in N_i$ (by Exercise 5). Then by the exactness at N_i, there exists $x_i \in M_i$ s.t. $\varphi_i(x_i) = y_i$. Let $x := \mu_i(x_i) \in M$, then $\varphi(x) = \varphi \circ \mu_i(x_i) = \nu_i \circ \varphi_i(x_i) = \nu_i(y_i) = y$, hence the exactness at N follows. □

Exercise 10 ([AM69]-exr-2.20)**.** show that

$$\varinjlim_i (M_i \otimes N) \cong \left(\varinjlim_i M_i \right) \otimes N$$

i.e. tensor product commutes with direct limits.

Proof. It is known from category theory that a right exact functor commutes with colimits [TODO: prove this], but we write some concrete nonsense here anyway.

Suffices to show the universal property of direct limits holds for $\left(\varinjlim_i M_i\right) \otimes N$ with respect to the direct system $(M_i \otimes N, \mu_{ij} \otimes \mathrm{id}_N)_{i,j\in I}$.

First, determine the canonical inclusions. Define $M_i \otimes N \to \left(\varinjlim_i M_i\right) \otimes N$ by $\mu_i \otimes \mathrm{id}_N$, i.e. sending $m_i \otimes n \mapsto \mu_i(m_i) \otimes n$ for $m_i \in M_i$, $n \in N$.

Next, verify the universal property. Say we have maps $\alpha_i : M_i \otimes N \to P$ satisfying the compatibility condition $\alpha_i = \alpha_j \circ (\mu_{ij} \otimes \mathrm{id}_N)$ for all $i \leq j$. To define a map $\bar{\alpha}_i : \left(\varinjlim_i M_i\right) \otimes N \to P$, it suffices to give a map $\varinjlim_i M_i \to \mathrm{Hom}_A(N, P)$. This in turn require us to define maps $M_i \to \mathrm{Hom}_A(N, P)$, and we define it as $m_i \mapsto (n \mapsto \alpha_i(m_i \otimes n))$. One verifies they commutes with transition maps μ_{ij} thanks to the compatibility condition $\alpha_i = \alpha_j \circ (\mu_{ij} \otimes \mathrm{id}_N)$, thus the map $\varinjlim_i M_i \to \mathrm{Hom}_A(N, P)$ is well-defined, an in turn $\bar{\alpha}_i : \left(\varinjlim_i M_i\right) \otimes N \to P$ is defined.

Finally, note that this choice of definition satisfies, and is unique to satisfy the universal property condition $\bar{\alpha}_i \circ (\mu_i \otimes \mathrm{id}_N) = \alpha_i$. Hence we have verified the universal property of $\left(\varinjlim_i M_i\right) \otimes N$ (with the defined canonical inclusions) w.r.t. the direct system $(M_i \otimes N, \mu_{ij} \otimes \mathrm{id}_N)_{i,j\in I}$. □

Exercise 11 ([AM69]-exr-2.21)**.** $(A_i)_{i\in I}$ a family of rings indexed by a directed set I. For each $i \le j$, let $\alpha_{ij} : A_i \to A_j$ be a ring homomorphism such that $\alpha_{ii} = \mathrm{id}_{A_i}$ and $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ for all $i \le j \le k$. As $\mathbb{Z}$-module we may view them as a direct system and obtain their direct limit $A := \varinjlim_i A_i$. Show that A has a natural ring structure such that the canonical maps $\alpha_i : A_i \to A$ are ring homomorphisms. This defines the direct limit in the category of rings.

In addition, show that $A = 0$ implies $A_i = 0$ for some i.

Proof. We define the multiplication on A as follows. For any $x, y \in A$, say $x = \alpha_i(x_i)$ and $y = \alpha_j(y_j)$ for some $i, j \in I$ and $x_i \in A_i$, $y_j \in A_j$ (by Exercise 5). Let $k \in I$ be an upper bound of i, j. Define $x \cdot y := \alpha_k(\alpha_{ik}(x_i) \cdot \alpha_{jk}(y_j))$. We verify that this definition is independent of the choice of representatives x_i, y_j and the choice of upper bound k.

If we have another choice of representatives $x_{i'} \in A_{i'}$, $y_{j'} \in A_{j'}$ for some $i', j' \in I$ with $x = \alpha_{i'}(x_{i'})$ and $y = \alpha_{j'}(y_{j'})$. Let $k' \in I$ be an upper bound of i', j'. First we claim that there exist an $l \in I$ s.t. $\alpha_{il}(x_i) = \alpha_{il}(x_{i'})$. Indeed, since $\alpha_k(\alpha_{ik}(x_i)) = x = \alpha_k(\alpha_{i'k}(x_{i'}))$, by Exercise 5 there exists an upper bound l of k, i' s.t. $\alpha_{i'l}(x_{i'}) = \alpha_{il}(x_i)$. Similarly, we can find an upper bound s.t. x_j and $x_{j'}$ agree after applying the transition map to that upper bound. WLOG, still use the letter l be an upper bound of these two upper bounds. Then

$$\begin{aligned}
\alpha_k(\alpha_{ik}(x_i) \cdot \alpha_{jk}(y_j)) &= \alpha_l(\alpha_{kl}(\alpha_{ik}(x_i) \cdot \alpha_{jk}(y_j))) \\
&= \alpha_l(\alpha_{il}(x_i) \cdot \alpha_{jl}(y_j)) \\
&= \alpha_l(\alpha_{i'l}(x_{i'}) \cdot \alpha_{j'l}(y_{j'})) \\
&= \alpha_{k'}(\alpha_{i'k'}(x_{i'}) \cdot \alpha_{j'k'}(y_{j'}))
\end{aligned}$$

Thus the definition is independent of the choice of representatives and upper bounds.

α_i is a ring homomorphism is straightforward to verify.

if $A = 0$, then for any $i \in I$, $\alpha_i(1_{A_i}) = 0$ in A. By Exercise 5, there exists an upper bound $j \in I$ of i s.t. $\alpha_{ij}(1_{A_i}) = 0$ in A_j. But the ring homomorphism α_{ij} must preserve identity, thus $A_j = 0$. □

Exercise 12 ([AM69]-exr-2.22)**.** (A_i, α_{ij}) direct system of rings. Let $\mathfrak{R}_i$ be the nilradical of A_i. Show that $\varinjlim_i \mathfrak{R}_i$ is the nilradical of $\varinjlim_i A_i$.

As a corollary, if each A_i is an integral domain, so is $\varinjlim A_i$.

Proof. For $i \in I$, $a_i \in A_i$, TFAE:

- $\alpha_i(a_i) \in \mathfrak{R}(\varinjlim_j A_j)$
- Exists some n s.t. $\alpha_i(a_i^n) = 0$
- Exists some n, $j \ge i$ s.t. $\alpha_{ij}(a_i^n) = 0$ (by Exercise 5)
- Exists some $j \ge i$ s.t. $\alpha_{ij}(a_i) \in \mathfrak{R}_j$
- $\alpha_i(a_i) \in \varinjlim_j \mathfrak{R}_j$

Thus $\mathfrak{R}(\varinjlim_j A_j) = \varinjlim_j \mathfrak{R}_j$ by Exercise 5. □

Exercise 13 ([AM69]-exr-2.23)**.** $(B_\lambda)_{\lambda\in\Lambda}$ a family of A-algebras. For each finite subset of Λ, let B_J be the tensor product of the B_λ for $\lambda \in J$. By inclusion of J, all B_J form a direct system. Let B denote its direct limit. Then B has a natrual A-algebra structure with $B_J \to B$ are A-algebra homomorphisms. This B is called the tensor product of the family $(B_\lambda)_{\lambda\in\Lambda}$ over A.

Proof. There is nothing to prove. □

Exercise 14 ([AM69]-exr-2.24). M is flat iff $\operatorname{Tor}_i(M, N) = 0$ for all N iff $\operatorname{Tor}_1(M, N) = 0$ for all N.

$i \implies ii$: Say

$$0 \to K \to P \to N$$

where P is projective. Write the long exact sequence:

$$\begin{array}{ccccccc} \operatorname{Tor}_2(K, M) & \longrightarrow & \operatorname{Tor}_2(P, M) & \longrightarrow & \operatorname{Tor}_2(N, M) & \longrightarrow & \\ \operatorname{Tor}_1(K, M) & \longrightarrow & \operatorname{Tor}_1(P, M) & \longrightarrow & \operatorname{Tor}_1(N, M) & \longrightarrow & \\ K \otimes M & \longrightarrow & P \otimes M & \longrightarrow & N \otimes M & \longrightarrow & 0 \end{array}$$

Since $-\otimes M$ is an exact functor. $K \otimes M \to P \otimes M$ is injective, i.e. $\operatorname{Tor}_1(N, M) \xrightarrow{0} K \otimes M$. Meanwhile, $\operatorname{Tor}_1(P, N) = 0$ since P is projective. This forces $\operatorname{Tor}_1(N, M) = 0$. Thus for all N, $\operatorname{Tor}^1(N, M) = 0$. In particular, $\operatorname{Tor}^1(K, M) = 0$. With $\operatorname{Tor}_2(P, M) = 0$, this fonces $\operatorname{Tor}^2(N, M) = 0$. Then inductively work upward gives $\operatorname{Tor}_i(N, M) = 0$ for all $i \geq 1$. □

$ii \implies iii$. Trivial. □

$iii \implies i$. The long exact sequence. □

Exercise 15 ([AM69]-exr-2.25).

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

is exact. N_3 is flat. Then N_1 is flat iff N_2 is flat.

Proof. Write the long exact sequence:

$$\begin{array}{ccccccc} \operatorname{Tor}_2(N_1, M) & \longrightarrow & \operatorname{Tor}_2(N_2, M) & \longrightarrow & \operatorname{Tor}_2(N_3, M) & \longrightarrow & \\ \operatorname{Tor}_1(N_1, M) & \longrightarrow & \operatorname{Tor}_1(N_2, M) & \longrightarrow & \operatorname{Tor}_1(N_3, M) & \longrightarrow & \\ N_1 \otimes M & \longrightarrow & N_2 \otimes M & \longrightarrow & N_3 \otimes M & \longrightarrow & 0 \end{array}$$

$\operatorname{Tor}_i(N_3, M) = 0$ by N_3 is flat.
If N_1 is flat. i.e. $\operatorname{Tor}_i(N_1, M) = 0$, then this force $\operatorname{Tor}_i(N_2, M) = 0$. i.e. N_2 is flat.
Similarly for the case that N_2 is flat. □

Exercise 16 ([AM69]-exr-2.26: Characterization of flatness via finitely generated ideals). N is an A-module. Then N is flat iff $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A.

$\implies$. Trivial (by Exercise 14). □

$\impliedby$. The hypothesis is that for all finitely generated ideals $\mathfrak{a}$ of A, the sequence

$$0 \longrightarrow \mathfrak{a} \otimes N \longrightarrow A \otimes N \longrightarrow A/\mathfrak{a} \otimes N \longrightarrow 0$$

is exact.

We first prove that for any ideal $\mathfrak{a}$ we have $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$. Consider the directed system of all finitely generated subideals of $\mathfrak{a}$, say $(\mathfrak{a}_i)_{i\in I}$. It's direct limit is $\mathfrak{a}$. Then we have an exact sequences of direct systems:

$$0 \longrightarrow (\mathfrak{a}_i)_{i\in I} \otimes N \longrightarrow \mathbf{A} \otimes N \longrightarrow (A/\mathfrak{a}_i)_{i\in I} \otimes N \longrightarrow 0$$

Taking direct limits, we get an exact sequence

$$0 \longrightarrow \mathfrak{a} \otimes N \longrightarrow A \otimes N \longrightarrow A/\mathfrak{a} \otimes N \longrightarrow 0$$

(by Exercise 9 exactness of taking direct limits, Exercise 10 commutativity of direct limit and tensor product). This shows that $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for any ideal $\mathfrak{a}$. i.e. $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$.

Note any cyclic A-modules are naturally isomorphic to $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A, thus $\operatorname{Tor}_1(M, N) = 0$ for any cyclic A-module M.

Next we show that for any finitely generated A-module M, $\operatorname{Tor}_1(M, N) = 0$. Consider a filtration of M:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

such that each successive quotient M_i/M_{i-1} is cyclic. Such a filtration exists by taking generators of M step by step. Consider the short exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$$

for each $1 \le i \le m$. Since M_i/M_{i-1} is cyclic, we have $\operatorname{Tor}_1(M_i/M_{i-1}, N) = 0$. By the proof of Exercise 15, we have $\operatorname{Tor}_1(M_i, N) \cong \operatorname{Tor}_1(M_{i-1}, N)$. Thus

$$\operatorname{Tor}_1(M, N) \cong \operatorname{Tor}_1(M_{m-1}, N) \cong \cdots \cong \operatorname{Tor}_1(M_1, N) \cong \operatorname{Tor}_1(M_0, N) = 0.$$

Finally, for any A-module M, we can write it as a direct limit of its finitely generated submodules, say $(M_i)_{i \in I}$. Then by a similar argument as above using Exercise 9 and Exercise 10 applied to the exact sequences

$$0 \longrightarrow \operatorname{Tor}_1(M_i, N) \longrightarrow K_i \otimes N \longrightarrow P_i \otimes N \longrightarrow M_i \otimes N \longrightarrow 0$$

where P_i is a free module, we know that for all A-module M, $\operatorname{Tor}_1(M, N) = 0$ and hence N is flat. □

Remark. The last step essentially shows that the Tor functor commutes with direct limits, i.e. for a directed system of A-modules $(M_i)_{i \in I}$,

$$\operatorname{Tor}_n \left(\varinjlim_i M_i, N \right) \cong \varinjlim_i \operatorname{Tor}_n (M_i, N)$$

for all $n \ge 0$. In particular, $n = 0$ is exactly the result of Exercise 10. [TODO] The proof above is just a vague sketch for now.

1.1 Extra explorations on Tor and flatness

The etymology of Tor: Quotient perspective

Say A is an integral domain and $\mathfrak{a}$ is a principal ideal generated by $a \in A$. Then from the short exact sequence

$$0 \longrightarrow A \xrightarrow{a\cdot -} A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

tensoring N we obtain the long exact sequence (leftmost item is zero since A is projective)

$$0 \longrightarrow \operatorname{Tor}_1(A/\mathfrak{a}, N) \longrightarrow N \xrightarrow{a\cdot -} N \longrightarrow N/\mathfrak{a}N \longrightarrow 0$$

We see that $\operatorname{Tor}_1(A/\mathfrak{a}, N) \cong \ker(a \cdot -) =: N[a]$, the a-torsion submodule of N. This explains the etymology of Tor.

The etymology of Tor: Localization perspective

Wikipedia if you want a quick reference for this part.

Another (somehow dual) way of understanding Tor is via localization. Say A_S is any localization of A with respect to a multiplicative set S. Consider the short exact sequence

$$0 \longrightarrow A \longrightarrow A_S \longrightarrow A_S/A \longrightarrow 0$$

Tensoring N to obtain the long exact sequence (leftmost item is zero since A_S as a localization of A is flat)

$$0 \longrightarrow \mathrm{Tor}_1(A_S/A, N) \longrightarrow N \longrightarrow S^{-1}N \longrightarrow (A_S/A) \otimes N \longrightarrow 0$$

We see that $\mathrm{Tor}_1(A_S/A, N) \cong \ker(N \to S^{-1}N)$. Thus $\mathrm{Tor}_1(A_S/A, N)$ measures those elements in N that collapse to zero upon localization $N \to S^{-1}N$. What are these elements? When A is an integral domain, they are precisely those elements $x \in N$ such that there exists $s \in S$ with $s \cdot x = 0$, i.e. the S-torsion submodule of N.

In particular, if we take the localization to be the field of fractions k, then we have realize the torsion submodule $T(M)$ of M as $\mathrm{Tor}_1(k/A, M)$, or the kernel of the localization map $M \to k \otimes M$. This gives another (probably better) explanation of the etymology of Tor.

Another example is to consider the localization at a prime ideal $\mathfrak{p}$, i.e. $S = A \setminus \mathfrak{p}$. In this case the terminology goes a little subtle… [TODO] continue this thought on PID.

It worths noting that the kernel of $k \otimes N \to (k/A) \otimes N$ captures the torsion-free part of N:

$$\ker(k \otimes N \to (k/A) \otimes N) \cong \mathrm{im}(N \to k \otimes N) \cong N/\ker(N \to k \otimes N) \cong N/T(N)$$

[TODO] continue this thought.

Characterizing flatness

We continue our discussion in Section 1.1. Recall Exercise 16 suggests that, to characterize the flatness of A-module N, it suffices to understand $\mathrm{Tor}_1(A/\mathfrak{a}, N)$ for all finitely generated ideals $\mathfrak{a}$ of A. So we ask: If A is not in general an integral domain, and $\mathfrak{a}$ is not in general principal, what is the meaning of $\mathrm{Tor}_1(A/\mathfrak{a}, N)$?

In this case, we can still use the short exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

to obtain the long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(A/\mathfrak{a}, N) \longrightarrow \mathfrak{a} \otimes N \xrightarrow{- \cdot -} N \longrightarrow A/\mathfrak{a}N \longrightarrow 0$$

From this we see that $\mathrm{Tor}_1(A/\mathfrak{a}, N) \cong \ker(\mathfrak{a} \otimes N \xrightarrow{- \cdot -} N)$, which can be viewed as the module of relations among the elements of $\mathfrak{a}$ when they act on N via the A-scalar multiplication. This can be interpreted as somehow a generalization of the torsion elements in the principal ideal case.

Combine this with Exercise 16, one may say that the flatness of A-module N is precisely characterized by the fact that any finite subset $\{a_1, \ldots, a_n\} \subseteq A$ acts on N without introducing any "$\mathfrak{a}$-torsion" or "$\mathfrak{a}$-relations", where $\mathfrak{a}$ is the ideal generated by $\{a_1, \ldots, a_n\}$. This intuition can be extremely useful, as the examples below will show.

Example 17 (Pure ideals)**.**

- For $n \geq 2$, $\mathbb{Z}/n\mathbb{Z}$ is not a flat $\mathbb{Z}$-module because it is not torsion-free.

  Recall that a finitely generated module over a PID is free iff it is torsion-free, and in particular flat.

- $2\mathbb{Z}/6\mathbb{Z}$ is a flat $\mathbb{Z}/6\mathbb{Z}$-module, not only by that it is projective (a direct summand of $\mathbb{Z}/6\mathbb{Z}$), but also because

  - 2 won't annihilate anything in $2\mathbb{Z}/6\mathbb{Z}$.
  - Dispite that 3 annihilates $2\mathbb{Z}/6\mathbb{Z}$, we have $3\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}/6\mathbb{Z}} 2\mathbb{Z}/6\mathbb{Z} = 0$. For example, $3 \otimes 2 = 3 \otimes 8 = 6 \otimes 4 = 0$. Thus 3-torsion won't give any nontrivial relation in this case.

  Blame $\mathbb{Z}/6\mathbb{Z}$ for this subtlety! Recall that over an integral domain, any flat module is torsion-free. But $\mathbb{Z}/6\mathbb{Z}$ is not an integral domain. In fact, any direct product of nontrivial rings ruins this property.

- Say k is a field, $A := k[x, y]$, $\mathfrak{a}$ is any nontrivial ideal of A. Then $A/\mathfrak{a}$ is not a flat A-module.

  In fact, pick any nonzero $a \in \mathfrak{a}$, then $a \otimes 1 \in \mathfrak{a} \otimes_A A/\mathfrak{a}$ is a nonzero element in the kernel of the multiplication map $\mathfrak{a} \otimes_A A/\mathfrak{a} \to A/\mathfrak{a}$.

  Note that $a \otimes 1$ being nonzero is guaranteed by the fact that A is an integral domain. Looks like arguments above can be generalized to any integral domain.

  Geometrically, this is an embedding of varieties $\operatorname{Spec} A/\mathfrak{a} \hookrightarrow \operatorname{Spec} A$.

Refer to the Stacks projects 04PQ for general discussions on when a quotient ring R/J is flat over R. Refer to 04PU, 04PV and 04PW for an algebraic geometry perspective.

Example 18 (Localizations)**.** Any localization $S^{-1}A$ of a ring A is always a flat A-module, because $S^{-1}A \otimes_A \mathfrak{a}$ is natrually isomorphic to $S^{-1}\mathfrak{a}$.

[TODO] Geometric examples.

Example 19 (A non-flat projection)**.** Let k be a field and $A = k[x, y]/(xy)$, $B = k[t]$. Define an B-module structure on A via the homomorphism $\varphi : B \to A$, $t \mapsto x$ (that is, identifying x in A with t). Then A is not a flat B-module.

This is because, if we pick $\mathfrak{t} := (t) = tB$ an ideal of B, then y is a t-torsion element in A (by $t \cdot y = \varphi(t)y = xy = 0$). That is, $t \otimes_{\mathfrak{t}} y$ is a nonzero element in the kernel of the multiplication map $\mathfrak{t} \otimes_B A \to A$.

Geometrically, this corresponds to projecting the union of two coordinate axes $\operatorname{Spec} A$ onto the x-axis, i.e. $\operatorname{Spec} \varphi : (x, y) \mapsto x$. Picking $\mathfrak{t} = (t)$ corresponds to looking at the fibers around $t = 0$. Note that the fiber over the origin $(0, 0)$ is the entire y-axis, while fibers over other points are singletons. This "jumping" of fiber dimension is reflected algebraically by the non-flatness of B as an A-module.

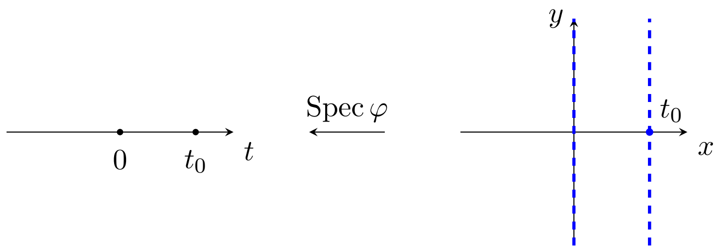

Figure 2: $\operatorname{Spec} \varphi$ and its fibers

Example 20 (A non-flat parametrization)**.** Let k be a field and $A = k[x, y]/(y^2 - x^3)$, $B = k[t]$. Define an A-module structure on B via the homomorphism $\varphi : A \to B$, $(x, y) \mapsto (t^2, t^3)$. Then B is not a flat A-module.

This is because, if we pick $\mathfrak{a} := (x, y)$ an ideal of A, then $x \cdot t = y \cdot 1_B$ becomes a nontrivial $\mathfrak{a}$-relation in B, i.e. $x \otimes_A t - y \otimes_A 1_B$ is a nonzero element in the kernel of the multiplication map $\mathfrak{a} \otimes_A B \to B$.

Geometrically, this corresponds to the fact that the parametrized curve $\operatorname{Spec} \varphi : t \mapsto (t^2, t^3)$ has a cusp at the origin. Picking $\mathfrak{a} = (x, y)$ corresponds to looking at the fibers of this map around the origin. [TODO] Understand this thoroughly.

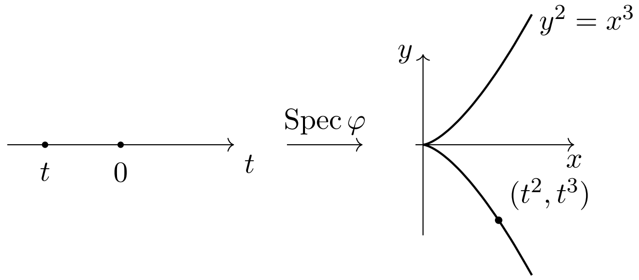

Figure 3: $\operatorname{Spec} \varphi$

Exercise 21 ([AM69]-exr-2.27: Absolute flatness)**.** A ring A is absolutely flat (or von Neumann regular) if every A-module is flat. Show that TFAE:

- i) A is absolutely flat;
- ii) every principal ideal of A is idempotent;
- iii) every finitely generated ideal of A is a direct summand of A.

Proof. i $\implies$ ii: Consider the commutative diagram

$$\begin{array}{ccc} (x) & \longrightarrow & (x) \otimes A/(x) \\ \downarrow & & \uparrow \\ A & \longrightarrow & A/(x) \end{array}$$

As inclusion, the left vertical arrow is injective, hence so do the right vertical arrow. As projection, the bottom horizontal arrow is surjective, hence so do the top horizontal arrow. Note that $(x) \to A \to A/(x)$ is zero, this forces the top horizontal arrow to be zero. But it is also surjective, hence $(x) \otimes A/(x) = 0$. Now note that $(x) \otimes A/(x) \cong (x)/(x^2)$, hence $(x) = (x^2)$.

ii $\implies$ iii: First we show that every principal ideal (x) is generated by an idempotent element e. Since $(x) = (x^2)$, we have $x = ax^2$, then $e := ax$ is idempotent and $(x) = (e)$. Next, for any two principal ideals (e) and (f) generated by idempotent elements e and f, we have $(e, f) = (e + f - ef)$ which is also generated by an idempotent element. Then inductively we know that any finitely generated ideal is generated by an idempotent element, hence is a direct summand of A since $A \cong A/(e) \oplus A/(1 - e) \cong (1 - e) \oplus (e)$.

iii $\implies$ i: By Exercise 16, to verify any A-module M is flat, it suffices to verify the flatness for for any finitely generated ideal $\mathfrak{a}$. Say $\mathfrak{a} \oplus \mathfrak{b} = A$, then

$$0 \longrightarrow \mathfrak{a} \longrightarrow A$$

tensoring M gives the sequence

$$0 \longrightarrow \mathfrak{a} \otimes M \longrightarrow A \otimes M = \mathfrak{a} \otimes M \oplus \mathfrak{b} \otimes M$$

hence is exact. ☐

Exercise 22 ([AM69]-exr-2.28)**.** A Boolean ring is absolutely flat.

Proof. Because in a Boolean ring, every element is idempotent. □

2 Chapter 3

Exercise 23 ([AM69]-exr-3.2)**.** Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Proof. Easily verify that S is an multiplicative set. It suffices to prove for all $\frac{a_1}{1+a_2} \in S^{-1}a$, $1 + \frac{a_1}{1+a_2} \cdot \frac{x}{1+a_3}$ is a unit in $S^{-1}A$ for all $\frac{x}{1+a_3} \in S^{-1}A$. The inverse $\frac{(1+a_2)(1+a_3)}{(1+a_2)(1+a_3)+a_1x}$ would do.

Proving [AM69]-cor-2.5: M f.g., $\mathfrak{a}$ is an ideal of A s.t. $\mathfrak{a}M = M$. Taking S^{-1} gives

$$\left(S^{-1}\mathfrak{a}\right)\left(S^{-1}M\right) = \left(S^{-1}\mathfrak{a}\right)M = S^{-1}M$$

By Nakayama's lemma, $S^{-1}M = 0$. This shows that $\exists s \in S$ s.t. $sM = 0$ (by [AM69]-exr-3.1). Say $s = 1 + a$. Done. □

Exercise 24 ([AM69]-exr-3.12: Torsion and torsion-free modules)**.** Let A be an integral domain, M be a finitely generated A-module, and $T(M)$ the torsion submodule of M. Show that

1. $T(M)$ is a indeed a submodule of M.
2. $M/T(M)$ is torsion-free.
3. Module homomorphism $f : M \to N$ maps $T(M)$ into $T(N)$.
4. Taking torsion is an left exact functor.
5. Let K be the field of fractions of A. Then $T(M)$ is exactly the kernel of the localization map $M \to K \otimes_A M$. Equivalently, there is a natural isomorphism $T(M) \cong \mathrm{Tor}_1^A(K/A, M)$.

Proof.

1. For any $x, y \in T(M)$, $\exists a, b \in A \setminus \{0\}$ s.t. $ax = 0$, $by = 0$. Then $ab(x+y) = abx + aby = 0 + 0 = 0$. Thus $x + y \in T(M)$. Similarly for scalar multiplication.

2. For any $x \in M$, if $\bar{x} \in M/T(M)$ is torsion, then $\exists a \in A \setminus \{0\}$ s.t. $a \cdot \bar{x} = \bar{0}$. This means $ax \in T(M)$, hence $x \in T(M)$ and $\bar{x} = \bar{0}$. Thus $M/T(M)$ is torsion-free.

3. For any $x \in T(M)$, $\exists a \in A \setminus \{0\}$ s.t. $ax = 0$. Then $af(x) = f(ax) = f(0) = 0$. Thus $f(x) \in T(N)$.

4. Say $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is exact. We need to show that $0 \to T(M_1) \xrightarrow{f} T(M_2) \xrightarrow{g} T(M_3)$ is exact.

First, $g \circ f = 0$ passes to the torsion submodules.

Secondly, f is injective on $T(M_1)$ by the injectivity of f on M_1.

Finally, for any $y \in T(M_2)$ with $g(y) = 0$, by the exactness of the original sequence, there exists $x \in M_1$ s.t. $f(x) = y$. Since y is torsion, $\exists b \in A \setminus \{0\}$ s.t. $b \cdot y = 0$. Then $0 = b \cdot y = b \cdot f(x) = f(b \cdot x)$, which implies $b \cdot x = 0$ by the injectivity of f. Thus $x \in T(M_1)$ and $f(x) = y$. This shows the exactness at $T(M_2)$.

5. See Section 1.1, where we systematically treat the etymology of Tor.

□

Exercise 25 ([AM69]-exr-3.13)**.** Taking torsion commutes with localization, i.e. for any multiplicative set S of A, $T(S^{-1}M) \cong S^{-1}T(M)$.

As a consequence, show that torsion-free is a local property.

Proof. Let k be the field of fractions of A. By Exercise 24, consider the exact sequence

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow k \otimes M$$

Taking localization S^{-1} gives the exact sequence

$$0 \longrightarrow S^{-1}T(M) \longrightarrow S^{-1}M \longrightarrow k \otimes M$$

Consider another exact sequence

$$0 \longrightarrow T(S^{-1}M) \longrightarrow S^{-1}M \longrightarrow k \otimes S^{-1}M \cong k \otimes M$$

Note that both $S^{-1}T(M)$ and $T(S^{-1}M)$ are realized as the kernels of the same map $S^{-1}M \to k \otimes M$. Thus they are naturally isomorphic.

For the consequence, say for all maximal ideals $\mathfrak{m}$ of A, $M_{\mathfrak{m}}$ is torsion-free. Then by the isomorphism above, $T(M)_{\mathfrak{m}} \cong T(M_{\mathfrak{m}}) = 0$. Being a zero module is a local property, thus M is torsion-free. □

Example 26 ([AM69]-exm-3.19: Support of a module)**.** Let A be a ring, M an A-module. Define the support of M to be the set $\operatorname{Supp} M$ of prime ideals $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$.

Show that

1. $M = 0$ iff $\operatorname{Supp} M = \varnothing$.
2. $V(\mathfrak{a}) = \operatorname{Supp}(A/\mathfrak{a})$.
3. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then $\operatorname{Supp} M_2 = \operatorname{Supp} M_1 \cup \operatorname{Supp} M_3$.
4. If $M = \sum_i M_i$, then $\operatorname{Supp} M = \bigcup_i \operatorname{Supp} M_i$.
5. If M is finitely generated, then $\operatorname{Supp} M = V(\operatorname{Ann}(M))$ (and hence is a closed subset of $\operatorname{Spec} A$).
6. If M and N are finitely generated, then $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp} M \cap \operatorname{Supp} N$.
7. If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $\operatorname{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \operatorname{Ann}(M))$.
8. If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $\operatorname{Supp}(B \otimes_A M) = (\operatorname{Spec} f)^{-1}(\operatorname{Supp} M)$.

Proof.

1. If $M = 0$, then for all prime ideal $\mathfrak{p}$, $M_{\mathfrak{p}} = 0$. Conversely, if $\operatorname{Supp} M = \varnothing$, then for all prime ideal $\mathfrak{p}$, $M_{\mathfrak{p}} = 0$. This implies that for all $x \in M$, $\exists s \in A \setminus \mathfrak{p}$ s.t. $sx = 0$. Thus the annihilator ideal $\operatorname{Ann}(x)$ is not contained in any prime ideal, which forces $\operatorname{Ann}(x) = A$ and hence $x = 0$. Thus $M = 0$.

2. For any prime ideal $\mathfrak{p} \in V(\mathfrak{a})$, $\mathfrak{a} \subseteq \mathfrak{p}$. Thus the annihilated port of $A_{\mathfrak{p}}$ never touches $\mathfrak{a}$ and hence $(A/\mathfrak{a})_{\mathfrak{p}} \neq 0$. Conversely, for any prime ideal $\mathfrak{p} \notin V(\mathfrak{a})$, $\exists a \in \mathfrak{a} \setminus \mathfrak{p}$. Then $a/1$ is a unit in $A_{\mathfrak{p}}$, which forces $(A/\mathfrak{a})_{\mathfrak{p}} = 0$.

3. By the exactness of localization, we have the exact sequence

$$0 \longrightarrow (M_1)_{\mathfrak{p}} \longrightarrow (M_2)_{\mathfrak{p}} \longrightarrow (M_3)_{\mathfrak{p}} \longrightarrow 0$$

   for all prime ideal $\mathfrak{p}$. Thus $(M_2)_{\mathfrak{p}} \neq 0$ iff either $(M_1)_{\mathfrak{p}} \neq 0$ or $(M_3)_{\mathfrak{p}} \neq 0$.

4. Localization, as a special tensor functor, commutes with taking sums (the later is a colimit, or more concretely, a direct limit) [TODO: Detail this nonsense] (A concrete proof can be found in Exercise 10). Thus for all prime ideal $\mathfrak{p}$, we have

$$M_{\mathfrak{p}} = \left(\sum_i M_i \right)_{\mathfrak{p}} = \sum_i (M_i)_{\mathfrak{p}}$$

   Hence the conclusion follows.

5. Say $M = \sum_i M_i$ where each M_i is cyclic, i.e. $M_i \cong A/\operatorname{Ann}(m_i)$ for some $m_i \in M_i$. Then by 4, we have

$$\operatorname{Supp} M = \bigcup_i \operatorname{Supp} M_i = \bigcup_i V(\operatorname{Ann}(m_i)) = V\left(\bigcap_i \operatorname{Ann}(m_i) \right) = V(\operatorname{Ann}(M))$$

   Note that the second last equality holds by the finiteness of the generating set $\{m_i\}$.

6. Recall that localization commutes with tensor product, i.e. for all prime ideal $\mathfrak{p}$,

$$(M \otimes_A N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$$

   Thus by Exercise 2, $(M \otimes_A N)_{\mathfrak{p}} = 0$ iff both $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$. The conclusion follows.

7. We have

$$\begin{aligned} \operatorname{Supp}(M/\mathfrak{a}M) &= \operatorname{Supp}(A/\mathfrak{a} \otimes_A M) = \operatorname{Supp}(A/\mathfrak{a}) \cap \operatorname{Supp}(M) \\ &= V(\mathfrak{a}) \cap V(\operatorname{Ann}(M)) = V(\mathfrak{a} + \operatorname{Ann}(M)) \end{aligned}$$

8. Say $\mathfrak{q} \in \operatorname{Spec} B$, and denote $\mathfrak{p} := f^{-1}(\mathfrak{q})$, we are to show that $(B \otimes_A M)_{\mathfrak{q}} = 0$ iff $M_{\mathfrak{p}} = 0$. It's clear that if $M_{\mathfrak{p}} = 0$, then $(B \otimes_A M)_{\mathfrak{q}} = 0$ by the isomorphism

$$(B \otimes_A M)_{\mathfrak{q}} \cong B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$$

   Conversely, if $(B \otimes_A M)_{\mathfrak{q}} = 0$, by the isomorphism above we have $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$. Note that we can't use Exercise 2 since $B_{\mathfrak{p}}$ may not be finitely generated over $A_{\mathfrak{p}}$. However, tensoring with the residue field $A_{\mathfrak{p}}/\mathfrak{p}$, we still have $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \left(M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \right) = 0$, where $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a finite-dimensional vector space over the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, and the whole LHS is a finite-dimensional vector space over the field $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$. Thus it must be of zero dimension. In turn $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = 0$. By Nakayama's lemma, $M_{\mathfrak{p}} = 0$.

□

Remark. [TODO] Best appreciated in a geometric view. If B is an A-algebra defined by the ring map $\varphi : A \to B$, then $\operatorname{Supp}_A B$ is exactly the image of $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, since for any prime ideal $\mathfrak{p}$ of A, the fiber over $\mathfrak{p}$ is $\operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$, which is nonempty iff $B \otimes_A k(\mathfrak{p}) \neq 0$ iff $B_{\mathfrak{p}} \neq 0$.

Exercise 27 ([AM69]-exr-3.20)**.** $f : A \to B$ ring homomorphism. Show that

1. Every prime ideal of A is a contracted ideal iff $\operatorname{Spec} f$ is surjective.
2. Every prime ideal of B is an extended ideal implies that $\operatorname{Spec} f$ is injective.
3. The converse of (2) is not true in general.

Proof.

1. By definition, $\mathfrak{p} \in \operatorname{Spec} A$ is a contracted ideal iff $\mathfrak{p} = f^{-1}(\mathfrak{q})$ for some prime ideal $\mathfrak{q}$ of B, exactly the surjectivity of $\operatorname{Spec} f$.

2. Denote $\mathfrak{a}^{\mathrm{e}}$ for ideal extension of $\mathfrak{p}$ alongside f, and $\mathfrak{b}^{\mathrm{c}}$ for contraction. For any $\mathfrak{q} \in \operatorname{Spec} B$, we have $\mathfrak{q} = \mathfrak{a}^{\mathrm{e}}$ for some ideal $\mathfrak{a}$ of A. This makes $\mathfrak{q} \mapsto \mathfrak{q}^{\mathrm{ce}}$ an identity map on $\operatorname{Spec} B$ by the fact that $\mathfrak{q}^{\mathrm{ce}} = \mathfrak{a}^{\mathrm{ece}} = \mathfrak{a}^{\mathrm{e}} = \mathfrak{q}$. Hence $\operatorname{Spec} f : \mathfrak{q} \mapsto \mathfrak{q}^{\mathrm{c}}$ is injective.

3. Consider $f : \mathbb{Z} \to \mathbb{Q}$. $\operatorname{Spec} f$ is injective since $\mathbb{Q}$ is a field. However, the prime ideal (0) of $\mathbb{Q}$ cannot be extended from any ideal of $\mathbb{Z}$.

□

Remark. If f is an integral extension, then the converse of (2) holds true. (By the going-up / lying-over / incomparability theorems.)

Exercise 28 ([AM69]-exr-3.21)**.**

1. Let A be a ring, S a multiplicative set of A, $\varphi : A \to S^{-1}A$ the localization homomorphism. Show that the map $\operatorname{Spec} \varphi : \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is a homeomorphism onto its image in $X = \operatorname{Spec} A$. Denote this image by $S^{-1}X$.

   In particular, if $f \in A$, then the map $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is a homeomorphism onto the basic open subset $D_A(f)$ of X.

2. Let $f : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$. Identify $S^{-1}X \subseteq X$ and $S^{-1}Y \subseteq Y$. Show that $\operatorname{Spec}(S^{-1}f) : \operatorname{Spec} S^{-1}B \to \operatorname{Spec} S^{-1}A$ is the restriction of $\operatorname{Spec} f$ to $S^{-1}Y$, and that $S^{-1}Y = (\operatorname{Spec} f)^{-1}(S^{-1}X)$.

3. Let $\mathfrak{a}$ be an ideal of A and $\mathfrak{b} = \mathfrak{a}^{\mathrm{e}}$ its extension alongside $f : A \to B$. Let $\bar{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the induced homomorphism. Identify $\operatorname{Spec}(A/\mathfrak{a}) = V(\mathfrak{a}) \subseteq X$ and $\operatorname{Spec}(B/\mathfrak{b}) = V(\mathfrak{b}) \subseteq Y$, show that $\operatorname{Spec}(\bar{f})$ is the restriction of $\operatorname{Spec} f$ to $V(\mathfrak{b})$.

4. Let $\mathfrak{p} \in \operatorname{Spec} A$. Take $S = A \setminus \mathfrak{p}$ in (2) and reduce mod $S^{-1}\mathfrak{p}$ as in (3). Deduce that the subspace $(\operatorname{Spec} f)^{-1}(\mathfrak{p})$ of $\operatorname{Spec} B$ is homeomorphic to $\operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$, the fiber of $\operatorname{Spec} f$ over $\mathfrak{p}$.

Proof.

1. Continuity and onto-ness are clear. Injectivity follows from Exercise 28 and the fact that every prime ideal of $S^{-1}A$ is an extended ideal from A. It suffices to show that $\operatorname{Spec} \varphi$ is an open map onto its image.

   Fix some $f/s \in S^{-1}A$. Consider any contracted ideal $\mathfrak{p} \in S^{-1}X$. Then it can be contracted from $S^{-1}\mathfrak{p}$. Suppose $S^{-1}\mathfrak{p} \in D_{S^{-1}A}(f/s)$, i.e. $f/s \notin S^{-1}\mathfrak{p}$ or equivalently any $t \in S$ won't make $f \in (\mathfrak{p} : t)$. Note that $\mathfrak{p}$ is contracted, hence $(\mathfrak{p} : t) = \mathfrak{p}$. Thus $f \notin \mathfrak{p}$, i.e. $\mathfrak{p} \in D_A(f)$. Above arguments can be reversed, hence we have shown that

$$\operatorname{Spec} \varphi \left(D_{S^{-1}A}(f/s)\right) = D_A(f) \cap S^{-1}X$$

2. Apply the Spec functor to the commutative diagram gives Figure 4.

The equation $S^{-1}Y = (\operatorname{Spec} f)^{-1}(S^{-1}X)$ is given by the fact that

$$\begin{aligned} & \mathfrak{q} \in S^{-1}Y \\ \iff & f(S) \cap \mathfrak{q} = \varnothing \\ \iff & \forall s \in S,\ f(s) \notin \mathfrak{q} \\ \iff & \forall s \in S,\ s \notin f^{-1}(\mathfrak{q}) \\ \iff & S \cap f^{-1}(\mathfrak{q}) = \varnothing \\ \iff & f^{-1}(\mathfrak{q}) \in S^{-1}X \\ \iff & \mathfrak{q} \in (\operatorname{Spec} f)^{-1}(S^{-1}X) \end{aligned}$$

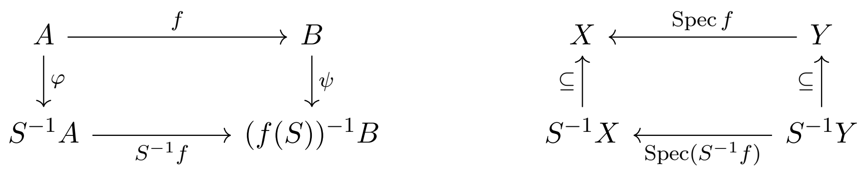

Figure 4

3. Similar to (2), see Figure 5.

Figure 5

4. Analyzing Figure 6 gives the desired homeomorphism.

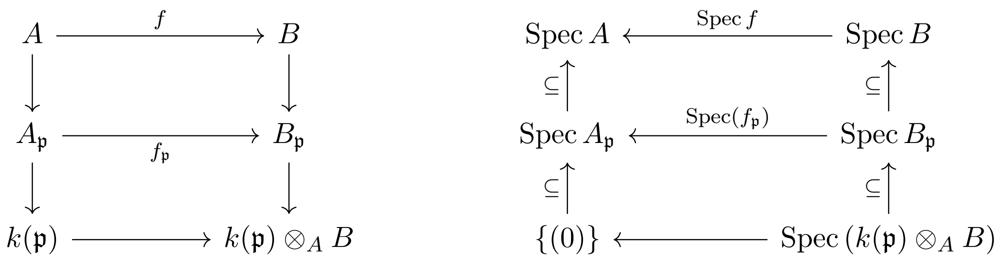

Figure 6

□

3 Chapter 4

Exercise 29 ([AM69]-exr-4.4)**.** In the polynomial ring $\mathfrak{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$-primary, but is not a power of $\mathfrak{m}$.

Proof. $\mathfrak{m}$ being maximal is clear. $\sqrt{\mathfrak{q}} = (2, t)$ is also clear. For any $ab \in \mathfrak{q}$, if $a \notin \mathfrak{m}$, then a is a odd integer. Hence $4 \mid b$ or $t \mid b$, i.e. $b \in \mathfrak{q}$, showing that $\mathfrak{q}$ is $\mathfrak{m}$-primary. Finally, note that $(2, t)^2 = (4, 2t, t^2) \neq (4, t) = \mathfrak{q}$. □

Exercise 30 ([AM69]-exr-4.5)**.** Consider the polynomial ring $K[x, y, z]$ where K is a field. Let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{m} = (x, y, z)$. Note that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals, and $\mathfrak{m}$ is a maximal ideal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2 = (x^2, xy, xz, yz)$. Show that $\mathfrak{a} := \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Identify the associated primes of $\mathfrak{a}$.

Proof. The mutual incomparability of $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}$ is clear. $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are minimal and $\mathfrak{m}$ is embedded. □

Exercise 31 ([AM69]-exr-4.6)**.** Let X be an infinite compact Hausdorff topological space, $C(X)$ be the ring of real-valued continuous functions on X (cf. [AM69]-exr-1.26). Is the zero ideal decomposable in this ring?

Proof. It can't be. If it is, recall that by the remark below [AM69, proposition 4.7], the set of zero-divisors of $C(X)$ is exactly the union of the associated primes. Consider the maximal ideals in these associated primes, they correspond to the points in X by [AM69, exr-1.26], say x_1, x_2, $\cdots$, x_n. Then every zero-divisor f must vanish at one of these points, i.e. $f(x_i) = 0$ for some $1 \leq i \leq n$. However, since X is infinite, we can always find some point $x \in X \setminus \{x_1, x_2, \cdots, x_n\}$. To reach a contradiction, we shall construct a continuous function g on X such that $g(x_i) \neq 0$ for all $1 \leq i \leq n$, but still a zero-divisor.

By the Hausdorff property, for each $1 \leq i \leq n$, we can find some open neighborhood U_i of x_i and some open neighborhood V_i of x such that $U_i \cap V_i = \varnothing$. Let $U := \bigcup_{i=1}^n U_i$, $V := \bigcap_{i=1}^n V_i$, then U and V are still open neighborhoods of $\{x_1, x_2, \cdots, x_n\}$ and x respectively, and $U \cap V = \varnothing$. By the Urysohn lemma, there exists some continuous function $g : X \to [0, 1]$ such that $g(x_i) = 1$ for all $1 \leq i \leq n$ and $g|_{X\setminus U} = 0$. It suffices to show that g is a zero-divisor. It's counterpart can be constructed in a symmetric way: By Urysohn lemma again, there exists a continuous function $h : X \to [0, 1]$ such that $h(x) = 1$ and $h|_{X\setminus V} = 0$. Then $gh = 0$, showing that g is a zero-divisor. □

Exercise 32 ([AM69]-exr-4.7)**.** A is a ring.

1. $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ alongside the inclusion $A \to A[x]$.
2. If $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.
3. If $\mathfrak{q}$ is $\mathfrak{p}$-primary in A, then $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary in $A[x]$.
4. If $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a primary decomposition of $\mathfrak{a}$ in A, then $\mathfrak{a}[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$ is a primary decomposition of $\mathfrak{a}[x]$ in $A[x]$.
5. If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

Proof.

1. Trivial.
2. $A[x]/p[x] \cong (A/p)[x]$ is an integral domain since A/p is so.
3. $A[x]/\mathfrak{q}[x] \cong (A/\mathfrak{q})[x]$. Any zero-divisor in $(A/\mathfrak{q})[x]$ must have all its coefficients being zero-divisors in $A/\mathfrak{q}$ by [AM69]-exr-1.2-iii, hence nilpotent since $\mathfrak{q}$ is $\mathfrak{p}$-primary. Thus this zero-divisor is nilpotent by [AM69]-exr-1.2-ii, and $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary. It's radical is at least $\mathfrak{p}[x]$ because $\mathfrak{q}$ can reach $\mathfrak{p}$ by taking radicals. Regarding that $\mathfrak{p}[x]$ is radical, we have $r(\mathfrak{q}[x]) = \mathfrak{p}[x]$.
4. There is nothing to prove.
5. There is nothing to prove.

□

Exercise 33 ([AM69]-exr-4.20: Primary decomposition of modules)**.** Let A be a ring, M be a fixed A-module, N a submodule of M. The radical of N in M is defined to be

$$r_M(N) = \{x \in A : x^q M \subseteq N \text{ for some } q > 0\}$$

Show that $r_M(N) = r(N : M) = r(\mathrm{Ann}(M/N))$.

Proof. Recall that $(N : M) = \{a \in A : aM \subseteq N\}$, hence the first equality is straightforward. The second equality follows from the fact that $\operatorname{Ann}(M/N) = (N : M)$. □

Remark. Analogous to [AM69]-exr-1.13, we have

- $r_M(N) \supseteq (N : M)$
- $r(r_M(N)) = r_M(N)$
- $r_M(N \cap P) = r_M(N) \cap r_M(P)$
- $r_M(N) = A$ iff $M = N$
- $r_M(N + P) \supseteq r(r_M(N) + r_M(P))$

Note that the last identity may not be an equality in general. For example, let $M = A \oplus A$, $N = A \oplus 0$, $P = 0 \oplus A$. Then $r_M(N) = r_M(P) = 0$, while $r_M(N + P) = r_M(M) = A$.

Exercise 34 ([AM69]-exr-4.21)**.** For any $x \in A$, φ_x denotes the multiplication endomorphism $M \to M$, $m \mapsto xm$. Then x is said to be a zero-divisor on M iff φ_x is not injective, is a nilpotent on M iff φ_x is nilpotent. A submodule Q of M is primary in M if $Q \neq M$ and every zero-divisor in M/Q is nilpotent.

Show that if Q is primary in M, then $(Q : M)$ is a primary ideal and hence $r_M(Q)$ is a prime ideal $\mathfrak{p}$. We say that Q is $\mathfrak{p}$-primary in M.

Proof. Recall that $(Q : M) = \operatorname{Ann}(M/Q)$. Note that $Q \neq M$ by definition of primary submodule. Say $ab \in (Q : M)$ for some $a, b \in A$.

In case that a is a zero-divisor on M/Q, then by the primary-ness of Q, a is nilpotent on M/Q and hence $a^n \in (Q : M)$ for some $n > 0$. If a is strong enough to kill M/Q, then $a \in (Q : M)$ and we are done. Otherwise b kills $aM/Q \neq 0$. Thus b is a zero-divisor on M/Q, hence nilpotent on M/Q, i.e. $b \in r(Q : M)$.

In case that a is not a zero-divisor on M/Q, then $aM/Q \neq 0$. Since ab kills M/Q, b must kill $aM/Q \neq 0$. Thus b is a zero-divisor on M/Q, hence nilpotent on M/Q, i.e. $b \in r(Q : M)$.

This shows that $(Q : M)$ is a primary ideal. □

Exercise 35 ([AM69]-exr-4.22)**.** A primary decomposition of N in M is a representation of N as an intersection

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

of primary submodules of M; it is a minimal primary decomposition if the ideals $\mathfrak{p}_i = r_M(Q_i)$ are all distinct and if none of the components Q_i can be omitted from the intersection.

Prove that the prime ideals $\mathfrak{p}_i$ depend only on N (and M), not on the particular minimal primary decomposition of N in M.

Proof. Note that $(N : M) = \bigcap_{i=1}^n (Q_i : M)$. By Exercise 34, each $(Q_i : M)$ is a $\mathfrak{p}_i$-primary ideal. Thus by the uniqueness of primary decomposition of ideals, the set $\{\mathfrak{p}_i\}$ is uniquely determined by $(N : M)$ and hence by N and M. □

4 Chapter 5

Exercise 36 ([AM69]-exr-5.1)**.** Let $f : A \to B$ be an integral homomorphism of rings. Show that $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed map, i.e. the image of a closed set is closed.

Proof. Let $V(\mathfrak{b}) \subseteq \operatorname{Spec} B$ be a closed set for some ideal $\mathfrak{b}$ of B. We are to show that $f^{-1}(V(\mathfrak{b}))$ is closed in $\operatorname{Spec} A$. Note that

$$f^{-1}(V(\mathfrak{b})) = \{\mathfrak{p} \in \operatorname{Spec} A : \exists \mathfrak{q} \in V(\mathfrak{b}),\ f^{-1}(\mathfrak{q}) = \mathfrak{p}\}$$

while

$$V(f^{-1}(\mathfrak{b})) = \{\mathfrak{p} \in \operatorname{Spec} A : f^{-1}(\mathfrak{b}) \subseteq \mathfrak{p}\}$$

Thus it suffices to show the two sets above are equal.

Write $f : A \to f(A) \subseteq B$, the former is surjective and the later is an integral extension. We have the ideal one-to-one correspondence between all ideals containing $\ker f$ of A, and all ideals of $f(A)$. By [AM69, theorem 5.10], every prime ideals of $f(A)$ can be contracted from some prime ideal of B. To summarize, every prime ideal of A containing $\ker f$ can be contracted from some prime ideal of B. We call this the lying-over theorem.

"$\subseteq$": For any $\mathfrak{p} \in f^{-1}(V(\mathfrak{b}))$, there exists some $\mathfrak{q} \in V(\mathfrak{b})$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Thus $f^{-1}(\mathfrak{b}) \subseteq f^{-1}(\mathfrak{q}) = \mathfrak{p}$, i.e. $\mathfrak{p} \in V(f^{-1}(\mathfrak{b}))$.

"$\supseteq$": For any $\mathfrak{p} \in V(f^{-1}(\mathfrak{b}))$, we have $\ker f \subseteq f^{-1}(\mathfrak{b}) \subseteq \mathfrak{p}$. By the lying-over theorem, there exists some prime ideal $\mathfrak{q}$ of B such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Note that $\mathfrak{b} \subseteq \mathfrak{q}$ since $f^{-1}(\mathfrak{b}) \subseteq \mathfrak{p} = f^{-1}(\mathfrak{q})$. Thus $\mathfrak{q} \in V(\mathfrak{b})$ and hence $\mathfrak{p} \in f^{-1}(V(\mathfrak{b}))$. □

Exercise 37 ([AM69]-exr-5.2)**.** Let $A \subseteq B$ is an integral extension of rings. Let $f \to \Omega$ be a homomorphism of A into an algebraically closed field Ω. Show that there exists an extension of f to a homomorphism $g : B \to \Omega$.

Proof. By Zorn's lemma it suffices to extend f to any intermediate ring $A[x]$ where $x \in B$. Note that x is integral over A, hence there exists some monic polynomial $p(t) \in A[t]$ such that $p(x) = 0$. Applying f to the coefficients of $p(t)$ gives a polynomial $f(p)(t) \in \Omega[t]$. Since Ω is algebraically closed, there exists some $a \in \Omega$ such that $f(p)(a) = 0$. We extend f to $g : A[x] \to \Omega$ by sending $x \mapsto a$. It's straightforward to verify that g is a well-defined ring homomorphism. □

Exercise 38 ([AM69]-exr-5.3)**.** The A-algebra functor $- \otimes_A C$ preserves integrality of A-algebra homomorphisms, i.e. if $f : B_1 \to B_2$ is an integral homomorphism of A-algebras, then so is $f \otimes \mathrm{id}_C : B_1 \otimes_A C \to B_2 \otimes_A C$ for any A-algebra C.

Proof. We are to show that any finite sum $\sum_i b_i \otimes c_i \in B_2 \otimes_A C$ for some $b_i \in B_2$ and $c_i \in C$ is integral over $B_1 \otimes_A C$. Since the integral closure is closed under addition, it suffices to show any $b \otimes c \in B_2 \otimes_A C$ is integral over $B_1 \otimes_A C$. By the integrality of f, there exists some monic polynomial $t^n + \sum_{i=0}^{n-1} a_i t^i \in B_1[t]$ such that $p(b) = 0$. Now

$$(b \otimes c)^n + \sum_{i=0}^{n-1} (a_i \otimes 1)(b \otimes c)^i = \left(b^n + \sum_{i=0}^{n-1} a_i b^i \right) \otimes c^n = 0$$

□

Exercise 39 ([AM69]-exr-5.6)**.** Let $B_1, \ldots, B_n$ be integral A-algebras. Show that $\prod_{i=1}^n B_i$ is an integral A-algebra.

Proof. By that integral closure is closed under finite sums, it suffices to show that any element of the form $(0, \ldots, 0, b, 0, \ldots, 0) \in \prod_{i=1}^n B_i$ is integral over A. Say b vanishes in some monic polynomial $f \in B_i[t]$. Then $(0, \ldots, 0, b, 0, \ldots, 0)$ vanishes in the same polynomial, where each coefficient is embedded via the canonical homomorhism $A \to \prod_{i=1}^n B_i$. □

5 Chapter 7

Exercise 40. TFAE for a Noetherian local ring $(A, \mathfrak{m}, k)$ and an A-module M:

1. M is free.
2. M is flat.

3. $\mathfrak{m} \otimes M \to A \otimes M$ is injective.
4. $\mathrm{Tor}_1^A(k, M) = 0$.

Proof.

(1) $\implies$ (2) is by that free modules are projective (by pulling back basis) and projective modules are flat (by the currification property of tensor functor).

(2) $\iff$ (3) is by the definition of flatness.

(3) $\implies$ (4) is by the long exact sequence of Tor alongside the short exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$$

It remains to show that (4) $\implies$ (1). Cosider the exact sequence

$$0 \longrightarrow Q \longrightarrow A^m \longrightarrow M \longrightarrow 0$$

where m is the minimal number of generators of M. By Nakayama's lemma, such number is also the dimension of $k \otimes M$ as a k-vector space. Applying $k \otimes -$ gives the long exact sequence

$$\mathrm{Tor}_1^A(k, M) = 0 \longrightarrow k \otimes Q \longrightarrow k^m \longrightarrow k^m \longrightarrow 0$$

Hence $k \otimes Q = 0$. By Nakayama's lemma again, we have $Q = 0$ and $M \cong A^m$ is free. □

6 Chapter 9

Exercise 41 ([AM69]-exr-9.9: Chinese Remainder Theorem over Dedekind domains)**.** Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $x_1, \ldots, x_n$ be elements in a Dedekind domain A. Then the system of congruences $x \equiv x_i \pmod{\mathfrak{a}_i}$ for $i = 1, \ldots, n$ has a solution $x \in A$ iff for all i, j, $x_i \equiv x_j \pmod{\mathfrak{a}_i + \mathfrak{a}_j}$.

Proof. Note that this is equivalent of the exactness of the following sequence:

$$A \longrightarrow \bigoplus_{i=1}^{n} A/\mathfrak{a}_i \longrightarrow \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j)$$

where the map on the left is the natural projection, and the map on the right sends $(\bar{x}_1, \ldots, \bar{x}_n)$ to $(\bar{x}_i - \bar{x}_j)_{i<j}$. The exactness is a local property, hence it suffices to assume A is a DVR since Dedekind domains are locally DVRs. In this case, each ideal $\mathfrak{a}_i$ is of the form (m^{k_i}) for some $k_i \geq 0$, where (m) is the unique maximal ideal of A. WLOG we assume $k_1 \leq k_2 \leq \cdots \leq k_n$. Then for any $i < j$, we have $\mathfrak{a}_i + \mathfrak{a}_j = (m^{k_i})$. Thus the exactness follows from the exactness of

$$A \longrightarrow \bigoplus_{i=1}^{n} A/(m^{k_i}) \longrightarrow \bigoplus_{i<j} A/(m^{k_i})$$

Say $(\bar{x}_1, \ldots, \bar{x}_n)$ lies in the kernel of the right map. Then for any $i < j$, $m^{k_i} \mid x_i - x_j$. Let $\delta_{i,j}$ be its divisor. We construct the preimage x as follows:

$$x = x_1 + \delta_{1,2} m^{k_1} + \delta_{2,3} m^{k_2} + \cdots + \delta_{n-1,n} m^{k_n}$$

It's strightforward to verify that it satisfy the congruence relation $x \equiv x_i \pmod{m^{k_i}}$. □

7 Chapter 11

Exercise 42 ([AM69]-exr-11.1)**.** Let $f \in k[x_1, \ldots, x_r]$ be an irreducible polynomial over an algebraically closed field k. A point P on the variety $f(x) = 0$ is non-singular iff not all the partial derivatives $\frac{\partial f}{\partial x_i}$ vanish at P. Let $A = k[x_1, \ldots, x_n]/(f)$, and let $\mathfrak{m}$ be the maximal ideal of A corresponding to P. Show that P is non-singular iff the localization $A_\mathfrak{m}$ is a regular local ring.

Proof. WLOG we assume P is the origin. Write

$$f(x_1, \ldots, x_r) = \sum_{i=1}^{r} \frac{\partial f}{\partial x_i}\bigg|_P x_i + \text{higher order terms}$$

Note that via pulling pack from A to $k[x_1, \ldots, x_r]$, we have

$$\mathfrak{m}/\mathfrak{m}^2 \cong ((x_1, \ldots, x_r) + (f)) \,/\, \big((x_1, \ldots, x_r)^2 + (f)\big) = (x_1, \ldots, x_r)/ \left((x_1, \ldots, x_r)^2 + (\sum_{i=1}^{r} \frac{\partial f}{\partial x_i}\bigg|_P x_i) \right)$$

In case that all partial derivatives vanish at P, then the dimension of the RHS is r since $x_1, \ldots, x_r$ form a basis. Otherwise, a nontrivial linear combination of $x_1, \ldots, x_r$ lies in the denominator, hence the dimension of the RHS is $r - 1$. □

Exercise 43. Reformulate [AM69, theorem 11.1] in terms of the Grothendieck group $K(A_0)$ (cf. [AM69]-exr-7.25)

Solution. Given a Noetherian graded ring $A = \bigoplus_{n=0}^{+\infty} A_n$, an additive function λ over all finitely-generated A_0-modules. By the universal property of Grothendieck group, for any additive group G, λ induces a group homomorphism $\tilde{\lambda} : K(A_0) \to G$. Then

$$P(M, t) = \sum_{n=0}^{+\infty} \lambda(M_n)t^n = \sum_{n=0}^{+\infty} \tilde{\lambda}([M_n])t^n$$

is a rational function of the form $\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$, where $f(t) \in \mathbb{Z}[t]$ and k_i are positive integers.

[TODO] What's the significance of this reformulation?

Exercise 44 ([AM69]-exr-11.6)**.** Let A be a ring, not necessarily Noetherian. Show that

$$1 + \dim A \le \dim A[x] \le 1 + 2 \dim A$$

Proof. Note that for any maximal ideal $\mathfrak{m}$ of A, $\mathfrak{m}[x] + (x)$ is a maximal ideal of $A[x]$ because it is the kernel of $A[x] \to A \to A/\mathfrak{m}$. Now say we have an maximal chain of prime ideals in A of length $\dim A$. The top prime ideal must be an maximal ideal of A, say $\mathfrak{m}$. Lifting it to $A[x]$, we have a chain of prime ideals in $A[x]$ of length $\dim A$, with top prime ideal $\mathfrak{m}[x]$. Extend this chain by adding one more prime ideal $\mathfrak{m} + (x)$ on the top, giving a chain of length $1 + \dim A$. Thus $\dim A[x] \ge 1 + \dim A$.

Consider any chain of prime ideals in $A[x]$. By contracting to A, we get a chain of prime ideals in A. Between any two consecutive prime ideals in A, there can be at most two prime ideals in $A[x]$: The fiber over $\mathfrak{p} \in \operatorname{Spec} A$ is homeomorphic to the spectrum of $k(\mathfrak{p}) \otimes A[x] = k(\mathfrak{p})[x]$ by Exercise 28, which has dimension 1. Thus the length of the chain in $A[x]$ is at most $1 + 2 \dim A$. □

Exercise 45 ([AM69]-exr-11.7)**.** Let A be a Noetherian ring, then $\dim A[x] = 1 + \dim A$.

Proof. Say $\mathfrak{p}$ is a prime ideal of A of height m. Localizing at $\mathfrak{p}$ gives a Noetherian local ring $A_\mathfrak{p}$ of dimension m. By the characterization of dimension [AM69, theorem 11.14], there exists an ideal $\mathfrak{a} = (a_1, \dots, a_m)$ with $\mathfrak{p}$ as one of its minimal prime ideals.

Now consider the height of $\mathfrak{p}[x]$. By Exercise 32, it is still a minimal associated prime ideal of $\mathfrak{a}[x]$, the latter is still generated by m elements. Thus by [AM69, corollary 11.16], the height of $\mathfrak{p}[x]$ is at most m.

We also have that the height of $\mathfrak{p}[x]$ is at least m, since every chain of prime ideals in A can be lifted to a chain of prime ideals in $A[x]$ via the $\mathfrak{p}_i \mapsto \mathfrak{p}_i[x]$. So the height of $\mathfrak{p}[x]$ is exactly m.

Now, for any prime ideal $\mathfrak{q}$ of $A[x]$ whose contraction to A is $\mathfrak{p}$, since $\mathfrak{p}[x]$ is of height m and also contracts to $\mathfrak{p}$, by the same fiber argument in Exercise 44, the height of $\mathfrak{q}$ is at most $m+1$. Thus the dimension of $A[x]$ is at most $1 + \dim A$. Combining with the other direction of inequality in Exercise 44, we have $\dim A[x] = 1 + \dim A$. □

8 Extra

Exercise 46. Let $R := k[x_1, \dots, x_n]$, $I := (x_1, \dots, x_n)$ is an R-ideal. Show that

$$R(I) \cong k[x_1, \dots, x_n, y_1, \dots, y_n]/(x_i y_j - x_j y_i)$$

where $R(I)$ is the Rees algebra of I, that is,

$$R(I) := \bigoplus_{d=0}^{+\infty} I^d t^d \subseteq R[t]$$

Proof. Define a homomorphism

$$\varphi : k[x_1, \dots, x_n, y_1, \dots, y_n] \to R(I), \quad x_i \mapsto x_i,\ y_i \mapsto x_i t$$

It's clearly surjective. Note that

$$\varphi(x_i y_j - x_j y_i) = x_i x_j t - x_j x_i t = 0$$

Hence the ideal $(x_i y_j - x_j y_i)$ lies in the kernel of φ. It remains to show that the kernel is exactly this ideal.

If some polynomial $f(x_1, \dots, x_n, y_1, \dots, y_n)$ lies in the kernel, then $f(x_1, \dots, x_n, x_1 t, \dots, x_n t) = 0$ in $R(I)$. Say f

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i_1, \dots, i_n, j_1, \dots, j_n} a_{i_1, \dots, i_n, j_1, \dots, j_n} x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n}$$

$f(x_1, \dots, x_n, x_1 t, \dots, x_n t) = 0$ shows that

$$\sum_{i_1, \dots, i_n; \sum j_\bullet = d} a_{i_1, \dots, i_n, j_1, \dots, j_n} x_1^{i_1 + j_1} \dots x_n^{i_n + j_n} = 0 \tag{1}$$

and in $k[x_1, \dots, x_n, y_1, \dots, y_n]/(x_i y_j - x_j y_i)$, note that

$$x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} = x_1^{i'_1} \dots x_n^{i'_n} y_1^{j'_1} \dots y_n^{j'_n}$$

when $i_k + j_k = i'_k + j'_k$ for all $1 \le k \le n$. Thus collecting terms with the same $i_k + j_k$, and apply Equation 1, $f = 0$ in $k[x_1, \dots, x_n, y_1, \dots, y_n]/(x_i y_j - x_j y_i)$. □

References

[AM69] Atiyah, M. F. and Macdonald, I. G. *Introduction To Commutative Algebra.* Addison-Wesley Publishing Company, Inc., 1969.