# Classification of Quardratic Forms over $\mathbb Q$

sun123zxy

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<sup>&</sup>lt;sup>1</sup>Last modified on 2025-04-22.

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#### **Foreword**

- References relied on heavily:
  - J.P. Serre "A Course in Arithmetic" [Ser73]
  - Shiva Chidambaram, MIT18.782 Introduction to Arithmetic Geometry (Spring 2023) Lecture Notes
  - Arushi Gupta, Participant Papers of The University of Chicago Mathematics REU 2018, The p-adic Integers, Analytically and Algebraically
- May serve as a guidance of the first part of [Ser73]
- Assume familarity of quadratic residues and basic knowledge of p-adic numbers
- Skip most of the proofs
- Apology in advance for potential mistakes

#### Notations

- We denote by K an arbitrary field. All fields are assumed to be of characteristic  $\neq 2$ .
- ullet  $u_p:\mathbb{Q}_p o\mathbb{Z}$  being the p-adic valuation.
- $\left(\frac{a}{p}\right)$  being the Legendre symbol. a is understanded as  $p^{-\nu_p(a)}a \bmod p$  if  $a \in \mathbb{Q}_p$ . Define this similarly in  $\mathbb{F}_q$ .
- ullet Let  $f \oplus g$  denote the direct sum of two quadratic forms f and g.

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### Review: Quadratic Forms over $\mathbb{R}$

• A quadratic form  $f\colon V\to K$  may be identified by a symmetric matrix  $A\in M_n(K)$  by  $f(v)=v^{\rm T}Av$ .

Their equivalence is defined by *congruence*:  $A \sim B \iff A = Q^T B Q$ .

- Real symmetric matrices may be diagonalized orthogonally.
- Scale each eigenvalue by multiplying a square. Only their sign matters.
  - $\bullet$  the rank n, an invariant
  - the signature (r, s) := (#positive eigenvalues, #negative eigenvalues).
- Same rank and signature implies the equivalence.
- Sylvester's law of inertia: signature is also an invariant.

### Some Refinement

#### On an abitrary field K:

- All symmetric matrix is equivalent to a diagonal one.
  - ullet Pick a non-isotropic vector v (exists when the form is nonzero), its orthogonal complement is a hyperplane and does not include v. Change basis and do the induction.
- The rank is always a invariant. We may (and we shall always) reduce to classify the non-degenerate quadratic forms of rank n.
- The squares  $(K^{\times})^2$  give us the ability to scale. Knowledge of distribution of diagonal elements in  $K^{\times}/(K^{\times})^2$  suffices to show the equivalence<sup>2</sup>.
  - $\mathbb{C}^{\times}/(\mathbb{C}^{\times})^2 \cong \{1\}$ , suffices to classify by the rank.
  - $\bullet$   $\mathbb{R}^{\times}/(\mathbb{R}^{\times})^2\cong\{1,-1\}$ , signature is also needed.
  - ullet  $\mathbb{F}_q^{\times}/(\mathbb{F}_q^{\times})^2\cong\{1,a\}$ , where  $a\in\mathbb{F}_q$  is a quadratic nonresidue.
  - For  $\mathbb{Q}_p$  and  $\mathbb{Q}$ ?

<sup>&</sup>lt;sup>2</sup>Though working in the refined structure  $\{0\} \cup K^{\times}/(K^{\times})^2$  is probably a better idea if one wish to deal with the degenerate case in a uniform manner  $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^$ 

# Another Example: Quadratic Forms over $\mathbb{F}_q$

We classify the non-degenerate quadratic forms of rank n.

- Refined signature: counting nonzero quadratic residues and nonresidues. It may serve as a sufficient criterion of the equivalence.
- $\bullet$  But it's not an invariant.  $aX^2+aY^2\sim X^2+Y^2$  over  $\mathbb{F}_q.$ 
  - Do a change of basis X=sU+tV and Y=tU-sV. If we require  $aU^2+aV^2=X^2+Y^2$ , then  $s^2+t^2=a$ .
  - It always has a nonzero solution in  $\mathbb{F}_q$ :  $s^2$  and  $a-t^2$  have both (q+1)/2 possible values, thus must reach a common value.
- The discriminant  $d:=\left(\frac{\det(A)}{q}\right)\in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$  is an invariant and reveals the parity of the signature. It classifies the non-degenerate quadratic forms over  $\mathbb{F}_q$ .

Insight: Existence of nonzero solutions of the equation  $aX^2 + bY^2 = Z^2$  in K seems to be of great importantance.

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# Quadratic Spaces

The structure of a quadratic space, i.e. vector space equipped with a symmetric bilinear form, is much more subtle than its positive-definite counterpart over  $\mathbb{R}$  or  $\mathbb{C}$ . For example, for a non-degenerate quadratic space V and a subspace U of V ([Ser73] p. 28, chap. 4, sec. 1.2):

- $U \cap U^{\perp} = \operatorname{rad}(U)$ , dim  $U + \operatorname{dim} U^{\perp} = \operatorname{dim} V$ ,  $(U^{\perp})^{\perp} = U$
- $\bullet \ \ U \oplus U^{\perp} = V \ \text{iff} \ \ U + U^{\perp} = V \ \text{iff} \ \operatorname{rad}(U) = 0$
- ullet It's much harder to show that an orthogonal basis of U expands to an orthogonal basis of V.

# Structure of Quadratic Spaces

We mention some results here without details.

Theorem 1.1 (Witt ([Ser73] p. 31, chap. 4, sec. 1.5, theorem 3))

Every injective metric-preserving map from a subspace U of a quadratic space V to another quadratic space W may be extended to a metric-preserving map from V to W.

Theorem 1.2 (Witt's cancellation ([Ser73] p. 34, chap. 4, sec. 1.6, theorem 4))

 $f_1 \oplus g_1 \sim f_2 \oplus g_2$  and  $g_1 \sim g_2$  implies  $f_1 \sim f_2$ .

### Theorem 1.3 (Witt's decomposition)

Every quadratic space V is a direct sum of:  $\mathrm{rad}(V)$ , an anisotropic quadratic space (i.e. its nonzero vectors has nonzero norms) and a split quadratic space (i.e.  $U=U^{\perp}$ , full of hyperbolas)

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# Another Invariant: The Range

On an abitrary field K, we say that a quadratic form f represents  $a \in K$  if there exists a nonzero  $v \in V$  such that f(v) = a.

- $\bullet$  The range of f,  $\operatorname{Im} f$ , is an invariant.
- It may be viewed in  $\{0\} \cup K^{\times}/(K^{\times})^2$ .
- Is it complete?

# Insights from the Range

### Proposition 1.1 (([Ser73] p. 33, chap. 4, sec. 1.6, corollary 1))

Let  $a \in K^{\times}$ . TFAE:

- f represents a
- $f \sim g \oplus (Z \mapsto aZ^2)$  where g is of rank  $\operatorname{rk} f 1$ .
- $f \oplus (Z \mapsto -aZ^2)$  represents 0.
- Insight from line 3: To understand the range, it suffices to examine when a quadratic form represents 0.
- Insight from line 2 (the common represented element method): Say  $f_1$ ,  $f_2$  are nozero and represent a common  $a \in K^{\times}$ . Reducing  $Z \mapsto aZ^2$ , if only  $g_1$  and  $g_2$  also share a common represented element...

### Insights from the Range

- Sadly, range is not always a complete invariant.
  - ullet Otherwise all indefinite quadratic forms over  ${\mathbb R}$  are equivalent, absurd.
- But we shall show that when  $K = \mathbb{Q}_p$  and moreover  $K = \mathbb{Q}$ , it plays a subtle role in the classification of quadratic forms. This requires a more precise characterization of the range.
- In fact, if only there are some simple invariants that can fully determines the range...

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### Global and Local Equivalence

- Fact: Field extensions preserve the equivalence of quadratic forms.
  - ullet Example: Equivalence classes are finer over  $\mathbb R$  than those over  $\mathbb C.$
- ullet  $\mathbb{Q}\hookrightarrow\mathbb{R},$  thus the rank and the signature are invariants. But we need more information to classify.
- Other field extension of  $\mathbb{Q}$ ?  $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$

Theorem 1.4 (Hasse-Minkowski ([Ser73] p. 41, chap. 4, sec. 3.1, theorem 8))

f represents 0 over  $\mathbb{Q}$  iff it represents 0 over  $\mathbb{R}$  and all  $\mathbb{Q}_p$ .

• To gain more invariants for  $\mathbb{Q}$  (especially those related to the range), let's classify quadratic forms over  $\mathbb{Q}_p$  first.

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# Structure of $\mathbb{Q}_p^{\times}$

- $\bullet \ \mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$  by collecting common powers of p
- $\bullet \ \mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p) \text{ by } a \mapsto a \bmod p$ 
  - It splits by the explicit construction of a primitive root of order p, via Hensel's lemma / Teichmüller lift  $\lim_{n\to\infty}g^{p^n}$ , where g is a primitive root of  $\mathbb{F}_n^{\times}$ .

# Structure of $1 + p\mathbb{Z}_p$ and the log / exp map

For  $p \neq 2$ ,  $\alpha \geq 1$  or p = 2,  $\alpha \geq 2$ :

$$1 + p^{\alpha} \mathbb{Z}_p \cong (p^{\alpha} \mathbb{Z}_p, +) \cong (\mathbb{Z}_p, +)$$
$$1 + p^{\alpha} a \mapsto \log(1 + p^{\alpha} a)$$

For  $p=2, \alpha=1$ ,

•

$$1 + 2\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \times (1 + 4\mathbb{Z}_2)$$

- by  $1 + 2a \mapsto a \mod 2$
- It splits by the explicit construction of a primitive root of order 2:  $(-1,-1,\dots)=\sum_{n=0}^{+\infty}2^n.$
- $(1+4\mathbb{Z}_2)\cong (4\mathbb{Z}_2,+)\cong (\mathbb{Z}_2,+)$  by the log map
- Thus

$$1 + 2\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}_2, +)$$



# Quadratic residues of $\mathbb{Q}_p$

#### For $p \neq 2$ :

- $\bullet \ \mathbb{Q}_p^{\times} \cong \mathbb{Z} \times \mathbb{F}_p^{\times} \times (\mathbb{Z}_p, +)$
- 2 is a unit in  $\mathbb{Z}_p$ . Thus  $a \in (\mathbb{Q}_p^{\times})^2$  iff  $\nu_p(a) \bmod 2 = 0$  and  $a \bmod p \in \mathbb{F}_p^{\times}$  is a quadratic residue.
- $\mathbb{Q}_p^{\times}/(\mathbb{Q}_p^{\times})^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , generated by p and a, where  $a \bmod p$  is a quadratic nonresidue

#### For p=2:

- $\mathbb{Q}_2^{\times} \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}_2, +)$
- Quadratic residues of  $(\mathbb{Z}_2,+)$  are  $(2\mathbb{Z}_2,+)$ , which pulls back to  $1+8\mathbb{Z}_2.$
- $a \in (\mathbb{Q}_2^{\times})^2$  iff  $\nu_2(a) \bmod 2 = 0$  and  $a \bmod 8 \equiv 1$ .
- $\mathbb{Q}_2^{\times}/(\mathbb{Q}_2^{\times})^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , generated by 2, 3 and 5.



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# The Hilbert Symbol

The Hilbert symbol over  $\mathbb{Q}_p$  is defined as:

$$\langle a,b\rangle := \begin{cases} 1 & \text{if } aX^2+bY^2=Z^2 \text{ has a nonzero solution in } \mathbb{Q}_p \\ -1 & \text{otherwise} \end{cases}$$

The symbol may be also viewed in  $\{0\} \cup \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$  or even more simply in  $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$  when working with non-degenerate forms.<sup>3</sup>

³Lots of the resources, even [Ser73], switch between these three views without enough warning. Sadly we shall also commit this ususal mild sin (and have already done to other innocent invariants such as the discriminant...)

# Properties of the Hilbert Symbol

- $\langle a, b \rangle = \langle b, a \rangle$  (symmetric)
- If  $\langle a_2, b \rangle = 1$ , then  $\langle a_1 a_2, b \rangle = \langle a_1, b \rangle$ 
  - In fact,  $\langle a_1 a_2, b \rangle = \langle a_1, b \rangle \langle a_2, b \rangle$  (multiplicatively bilinear)
- $\langle a, b \rangle = 1$  for all b iff  $a \in \mathbb{Q}_p^2$  (nondegenerate)
- the Hilbert symbol is a non-degenerate symmetric bilinear form of the  $\mathbb{F}_2$ -vector space  $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ 
  - This is a non-trivial result and is said to be, to some extent, a generalization of the law of quadratic reciprocity in local class field theory.
  - To show above over  $\mathbb{Q}_p$ , we develop an explicit formula for the Hilbert symbol.

# The Explicit Formula of the Hilbert Symbol

### Theorem 2.1 (([Ser73] p. 20, chap. 3, sec. 1.2, theorem 1))

Say  $a=p^{\alpha}u$  and  $b=p^{\beta}v$  are p-adic numbers where  $u,v\in\mathbb{Z}_p^{\times}$  , then

$$\langle a,b \rangle = (-1)^{\alpha \cdot \beta \cdot \frac{p-1}{2}} \left(\frac{u}{p}\right)^{\beta} \left(\frac{v}{p}\right)^{\alpha} \text{ if } p \neq 2$$

We omit the case p=2. It's a tedious modification of the above formula.

	0	1	a	p
0	1	1	1	1
1		1	1	1
a			1	-1
p				$(-1)^{\frac{p-1}{2}}$

Table 1: Table of Hilbert symbol,  $p \neq 2$ 

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#### The Hasse Invariant

Recall that we have reduced to work with non-degenerate diagonalized quadratic forms of rank n. Recall that the discriminant

$$d(f) = a_1 a_2 \dots a_n \in \mathbb{Q}_p^{\times} / (\mathbb{Q}_p^{\times})^2$$

is an invariant.

- Define the Hasse invariant  $\varepsilon(f) := \prod_{1 \le i \le j \le n} \langle a_i, a_j \rangle$
- It is an invariant:

$$\varepsilon(f) = \prod_{1 \le i < j \le n} \langle a_i, a_j \rangle = \varepsilon(f_1) \prod_{2 \le j \le n} \langle a_1, a_j \rangle = \varepsilon(f_1) \cdot \langle a_1, a_1 d(f) \rangle$$

Thus  $\varepsilon$  is preserved under *contiguous* change of orthogonal bases (fixes one of the vector of the basis)

• For  $n \ge 3$ , orthorgonal bases are transitive under contiguous change ([Ser73] p. 30, sec. 4.1.4, theorem 2)

# d and $\varepsilon$ Determine the Range

# Theorem 2.2 (([Ser73] p. 36, chap. 4, sec. 2.2, theorem 6))

For a non-degenerate quadratic form f of rank n over  $\mathbb{Q}_p$ , the range of f is determined by the discriminant d:=d(f) and the Hasse invariant  $\varepsilon:=\varepsilon(f)$ . Or, in detail, f represents 0 iff:

- For n = 2: d = -1
- For n=3:  $\langle -1, -d \rangle = \varepsilon$
- For n=4:  $d \neq 1$  or d=1 and  $\varepsilon = \langle -1, -1 \rangle$
- For n = 5: no conditions

Recall that f represents  $a\in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$  iff  $f\oplus (Z\mapsto -aZ^2)$  represents 0, thus above fully characterizes the range.

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# Classification of Quadratic Forms over $\mathbb{Q}_p$

# Theorem 2.3 (([Ser73] p. 39, chap. 4, sec. 2.3, theorem 7))

Two non-degenerate quadratic forms of rank n over  $\mathbb{Q}_p$  are equivalent iff they have the same discriminant d and Hasse invariant  $\varepsilon$ .

- f, g have same d and  $\varepsilon$ , thus have the same range. Say they both represent  $a \in \mathbb{Q}_p^{\times}/(\mathbb{Q}_p^{\times})^2$ .
- Then  $f \sim f_1 \oplus (Z \mapsto aZ^2)$ , where  $f_1$  is of rank n-1.
- d and  $\varepsilon$  of  $f_1$  can be determined:
  - $d(f_1) = ad(f)$
  - $\varepsilon(f_1) = \varepsilon(f) \cdot (a, ad(f))$  (shown when discussing the invariance of  $\varepsilon$ )
- The same for g. Thus  $f_1, g_1$  shares the same d and  $\varepsilon$  (thus also their range). QED by induction.



# Classification of Quadratic Forms over $\mathbb Q$

### Theorem 2.4 (([Ser73] p. 39, chap. 4, sec. 2.3, theorem 7))

Two non-degenerate quadratic forms of rank n over  $\mathbb{Q}_p$  are equivalent iff they are equivalent over  $\mathbb{R}$  and over each  $\mathbb{Q}_p$ .

- Say f, g are equivalent over each local field ( $\mathbb{Q}_p$  and  $\mathbb{R}$ ), thus they share the same range locally.
- $\bullet$  By Hasse-Minkowski theorem, they also share the same range globally over  $\mathbb{Q}.$
- Then  $f \sim f_1 \oplus (Z \mapsto aZ^2)$  globally, where  $f_1$  is of rank n-1. The same for g.
- ullet  $f_1\sim g_1$  locally by Witt's cancellation theorem. QED by induction.

#### **Problem Remains**

- Proof of the Hasse-Minkowski theorem
  - essentially needs some understanding of the global property of the Hilbert symbol, which we have not discussed (cf. [Ser73])
- Refine the theory for degenerate quadratic forms (relatively easy)
- Enumerate all the equivalence classes of quadratic forms over  $\mathbb{Q}_p$  and  $\mathbb{Q}$  (cf. [Ser73])
- To what extent can we use the common represented element method to classify quadratic forms over other fields?
- For which fields, the range of a quadratic form is a complete invariant? (At least  $\mathbb{R}$  fails.  $\mathbb{Q}$ ,  $\mathbb{Q}_p$ ?)
- What can we say about  $\mathbb{Q}^{\times}/(\mathbb{Q}^{\times})^2$ ?
- Classification of quadratic forms over commutative rings (e.g.  $\mathbb{Z}$ ,  $\mathbb{Z}/m\mathbb{Z}$ )

#### References I

[Ser73] Jean-Pierre Serre. A Course in Arithmetic. Vol. 7. Graduate Texts in Mathematics. New York, NY: Springer, 1973. ISBN: 978-0-387-90041-4 978-1-4684-9884-4. DOI: 10.1007/978-1-4684-9884-4.